Introductory Cosmology

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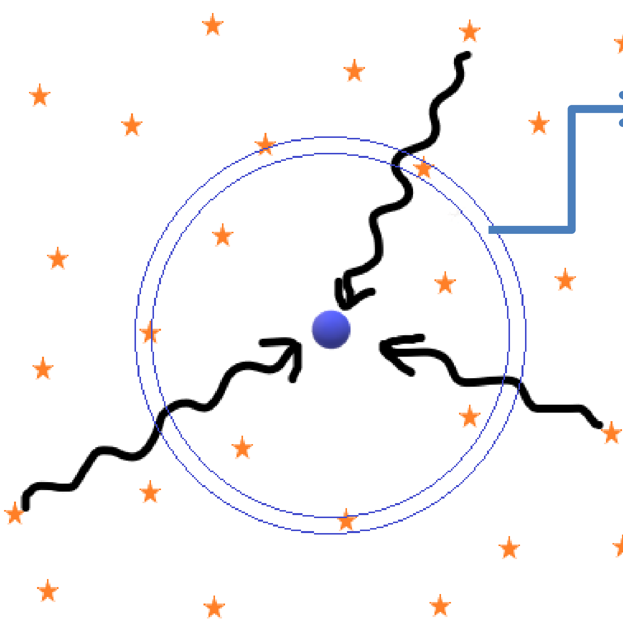
Piyabut Burikham

Problems with Newtonian Cosmology

- Gravity attracts, mass should clump, universe should be dynamical → infinite universe as solution, i.e. no center to collapse to!
- Olber paradox → infinite universe forever exists has infinite radiation flux at any point!!



Light flux received from this shell $= A\dfrac{L_0}{4\pi r^2}4\pi r^2 n\,dr$

L_0 = luminosity of each star, n = number density of star,

A = cross section of e.g. Earth

$$\text{Total flux} = A\int_0^{r_U}\frac{L_0}{4\pi r^2}4\pi r^2 n\,dr = nAL_0 r_U \rightarrow \infty$$

r_U = radius of Universe

Space is dark → Finite Universe with finite age!

- Infinite Universe existing ever forever must be utterly dark(run out of fuel), full of black holes.
- Finite Universe existing ever forever must have collapsed completely into single gigantic black hole.
- Alternatives emerge in General Relativity(GR);

  Closed, Flat, Open dynamical Universe

Hubble discoveries

- Using Cepheids' Period-Luminosity relation discovered by Henrietta Leavitt, establish that spiral "nebulae" are *spiral galaxies!* Hence Universe size is hundreds thousand times larger than believed at that time.
- (1929) Found linear relation between redshift&distance of far away galaxies,

$$z \equiv \frac{\lambda_O - \lambda_S}{\lambda_S} = \frac{\lambda_O}{\lambda_S} - 1, \quad \boxed{v(\simeq zc) = Kd}$$

- This is called Hubble law, will derive later from FLRW metric.

GR → spacetime lump = Universe

- Gravity = Spacetime curvature = κ energy&momentum density.
- Matter tells spacetime how to curve,
  Spacetime tells matter how to move.

$$\kappa = \frac{8\pi G}{c^4} = \text{unit conversion factor}$$

$$G_{ab} = R_{ab} - \tfrac{1}{2} R g_{ab} + \Lambda g_{ab} = \frac{8\pi G}{c^4} T_{ab}$$

Where *Ricci tensor* $R_{ab} = R^{c}{}_{acb}$ and the *Riemann curvature tensor* is given by

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}{}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma}$$

The *energy-momentum tensor* T_{ab} contains info of matter
And energy distribution.

$$\Gamma^{\mu}_{\alpha\beta} \equiv \frac{1}{2} g^{\mu\lambda}(\partial_{\alpha} g_{\beta\lambda} + \partial_{\beta} g_{\alpha\lambda} - \partial_{\lambda} g_{\alpha\beta})$$

Two more curvatures

- Ricci tensor: $$R_{ab} = R^{d}_{adb} = -R^{d}_{abd}\,,$$ $$R_{ab} = R_{ba}$$
- Ricci scalar: $$R = R^{a}_{a}$$
- *Ex*: sphere S^2, $$\{g_{ab}\} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \sin^2\theta \end{pmatrix}$$

(Use Mathematica to) compute

$$R^{\phi}_{\theta\phi\theta} = 1, R^{\theta}_{\phi\theta\phi} = \sin^2\theta,$$
$$R_{\theta\theta} = 1, R_{\phi\phi} = \sin^2\theta,$$
$$R = \frac{2}{a^2}$$

- Hyperbolic H^2, $$\sin \Rightarrow \sinh,$$ $$\mathrm{R} = -\frac{2}{a^2}$$

Cosmological principle(Newton&Einstein)

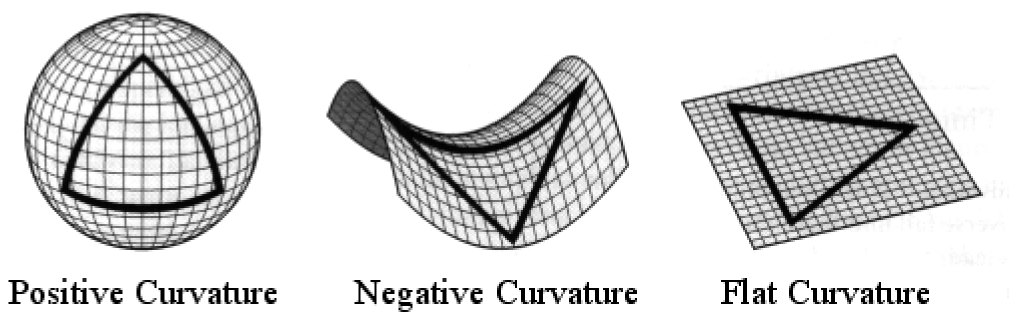
- Homogeneous universe: uniform and is the same everywhere
- Isotropic universe: is the same in every direction
- Homogeneity implies no special point in universe. Combined with Isotropy implies that there can't be special direction(*anisotropy*) and special observer in the universe.

FLRW metric ansatz

- Friedmann was the first (1922), most generic metric obeying cosmological principle

$$ds^2 = c^2 dt^2 - R^2(t)(f^2(r)dr^2 + r^2 d\Omega); \; d\Omega = d\theta^2 + \sin^2\theta d\phi^2$$

- From homogeneity, we demand that $R_{3D} = \dfrac{6k}{R^2(t)}$



$k > 0$ $\quad$ $k < 0$ $\quad$ $k = 0$

Picture from https://pages.uoregon.edu/jschombe/cosmo/lectures/lec15.html

3D shape of Universe

- (Use Mathematica to) compute

$$R_{3D} = \frac{1}{f^2(r) r^2 R^2(t)} \left(2(f^2 - 1) + 4r \frac{f'}{f} \right) = \frac{6k}{R^2(t)}$$

- We can solve to obtain $f(r) = \left(1 - kr^2\right)^{-1/2}$
- By redefining $r \to r\sqrt{k}, R(t) \to R(t)\sqrt{k},$

$$ds^2 = c^2 dt^2 - R^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega \right), k = 1, 0, -1$$

- Hubble law is natural.

$$\vec{r} = \varpi_r R(t), \vec{v} = \varpi_r \dot{R}(t) = \frac{\dot{R}(t)}{R(t)} \varpi_r R(t) \equiv H\vec{r},$$

Cosmological Principle obeyed:

$$\vec{v}_{12} = (\varpi_{r1} - \varpi_{r2}) \dot{R}(t) = H(\vec{r}_1 - \vec{r}_2) = H\vec{r}_{12}$$

Solve Einstein eqn. from this metric

$$G_{ab} = R_{ab} - \tfrac{1}{2} R g_{ab} = \frac{8\pi G}{c^4} T_{ab}$$

$$G_{00} = R_{00} - \frac{g_{00}}{2} R(icci) = 3\left(\frac{k}{R^2(t)} + H^2 \right), H = \frac{\dot{R}}{R}$$

$$G_{rr} = R_{rr} - \frac{g_{rr}}{2} R(icci) = -\frac{1}{1-kr^2}(1+\dot{R}^2 + 2R\ddot{R}),$$

Rests are reduntant to (rr) – component.

- Assume perfect fluid: $T^{ab} = \left(\rho + \frac{P}{c^2} \right) u^a u^b - P g^{ab},\ u^a = (c, \vec{0})$ in comoving frame
- Will use Mathematica to compute later.
- Note the relation in FLRW metric: $z = \frac{\lambda_O}{\lambda_S} - 1,\ \boxed{\frac{R_0}{R} = 1 + z}$

Einstein field eqn. of FLRW metric

$$(tt-\text{component});\quad \left(\frac{\dot{R}}{R}\right)^2+\frac{k}{R^2}=\frac{8}{3}\pi G\rho$$

Friedmann equation

$$(rr-\text{component}\&tt-\text{component});\quad \frac{\ddot{R}}{R}=-\frac{4}{3}\pi G(\rho+3P)$$

Acceleration equation

- Another redundant eqn. from

$$T^a_{b;\mathrm{a}}=0\Rightarrow\dot{\rho}=-\frac{3\dot{R}}{R}(\rho+P)\ ;\ \text{conservation eqn.}$$

- Observe that accelerated expansion requires $P<-\frac{\rho}{3}$ → Dark Energy

Linear equation of state

- To solve for 3 unknown $P, \rho, R(t)$, we still need what-so-called Equation of State (EoS), i.e., additional info about P, ρ
- For compact object, it could be polytrope $P = \rho^n$, or even

  $P = P(\mu, T), \rho = \rho(\mu, T)$
- But for cosmology, it suffices(?) to assume linear EoS $P = w\rho$
- Sub into conservation eqn. to obtain $\frac{\rho}{\rho_0} = \left(\frac{R_0}{R}\right)^{3(1+w)}$
- Sub into Friedmann eqn. to obtain

$$R(t) \sim t^{2/3(1+w)} \quad \text{for } w \neq -1,$$
$$R(t) \sim e^{Ht} \quad \text{for } w = -1$$

3 main eras of Universe

- radiation-dominated: $$w_r = \frac{1}{3}, \frac{\rho}{\rho_0} = \left(\frac{R_0}{R}\right)^4, R \sim t^{1/2}$$

  extremely hot so that most particles can treated relativistically.
- matter-dominated: $$w_m = 0, \frac{\rho}{\rho_0} = \left(\frac{R_0}{R}\right)^3, R \sim t^{2/3}$$

  cold enough to treat as non-relativistic particle → simplified to "dust".
- Dark energy dominated: $$w_{DE} \simeq -1, , \rho \simeq \rho_\Lambda, R \sim e^{H_\Lambda t}$$
  roughly 8 billion years ago until present, accelerated expansion of Universe.
- Density parameter: From Observations,

$$\Omega_i(t) \equiv \frac{\rho_i(t)}{\rho_c(t)} = \frac{8\pi G}{3H^2}\rho_i;\ \rho_c \equiv \text{critical density for flat Universe; } \Omega_{total} \simeq 1 (k \simeq 0, \text{flat})$$

$$\Omega_{DM,0} \simeq 0.26,\ \Omega_{b,0}, 0 \simeq 0.05,\ \Omega_{r,0} \simeq 10^{-5},\ \Omega_{DE,0} = 0.68 \text{ (present)}$$

Another era, curvature era

- Even though (non-normalized) *k is close to 0,* there is error bar, so there could be a slight *spatial curvature* in the Universe!
- Curvature effects emerge at later time:

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8}{3}\pi G\rho = \frac{8}{3}\pi G \sum_{i=b,m,r,DE,...} \rho_{i,0}\left(\frac{R_0}{R}\right)^{3(1+w)},$$

For $w > -\frac{1}{3}$, at large R, curvature term dominates.

- Non DE effects will fade away as Universe evolves. So *curvature* could dominate (or important) *between matter-era and DE-era* → One possible proposal for *Hubble tension* solution.

Friedmann eqn. in density parameters

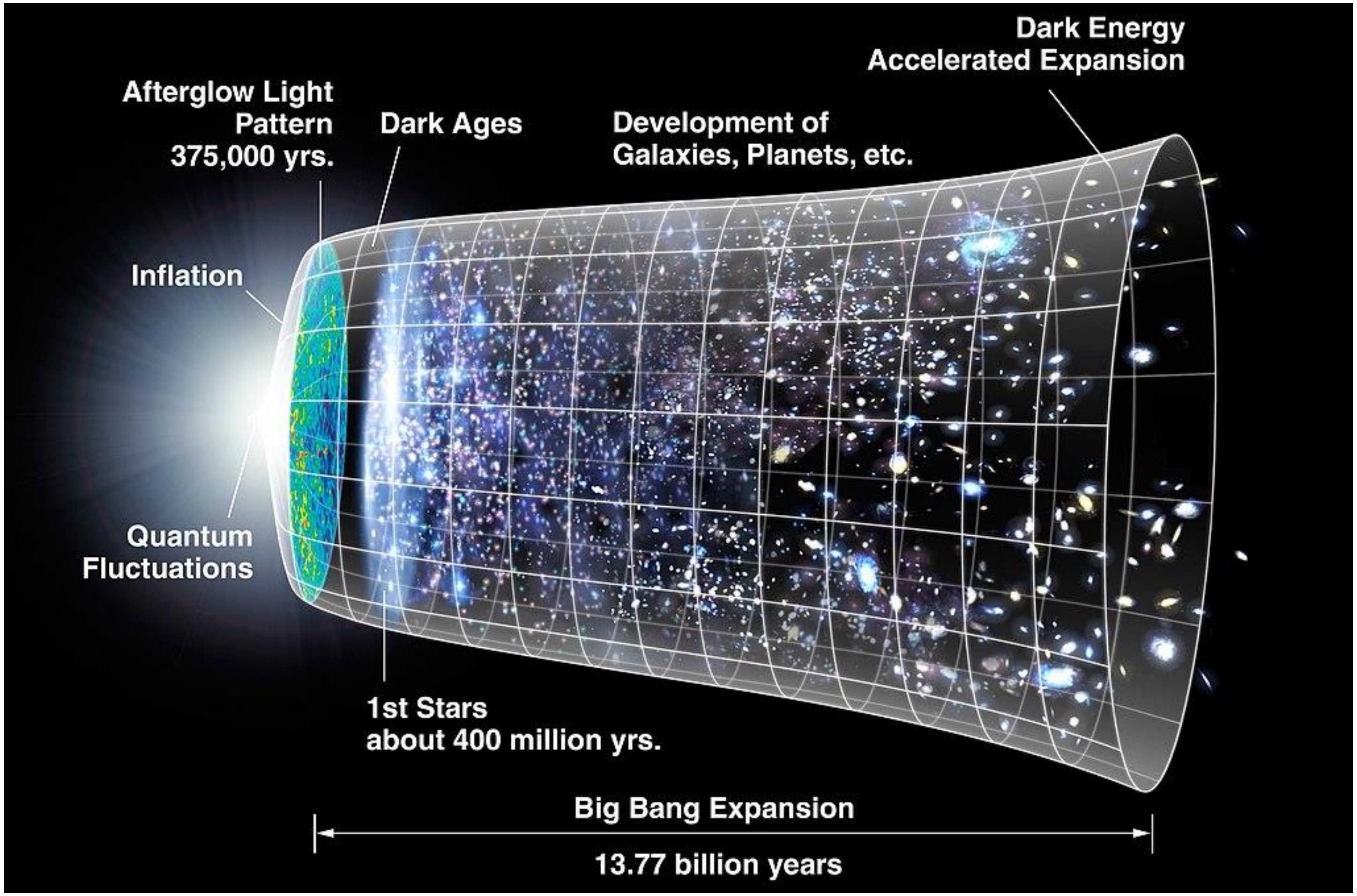
- Generalized form of Friedmann:

$$H^2 = H_0^2\left(\sum_{i=\ldots}\Omega_{i,0}R^{-3(1+w_i)} + (1-\Omega_0)R^{-2}\right) = H_0^2\left(\sum_{i=\ldots,k}\Omega_{i,0}R^{-3(1+w_i)}\right)$$

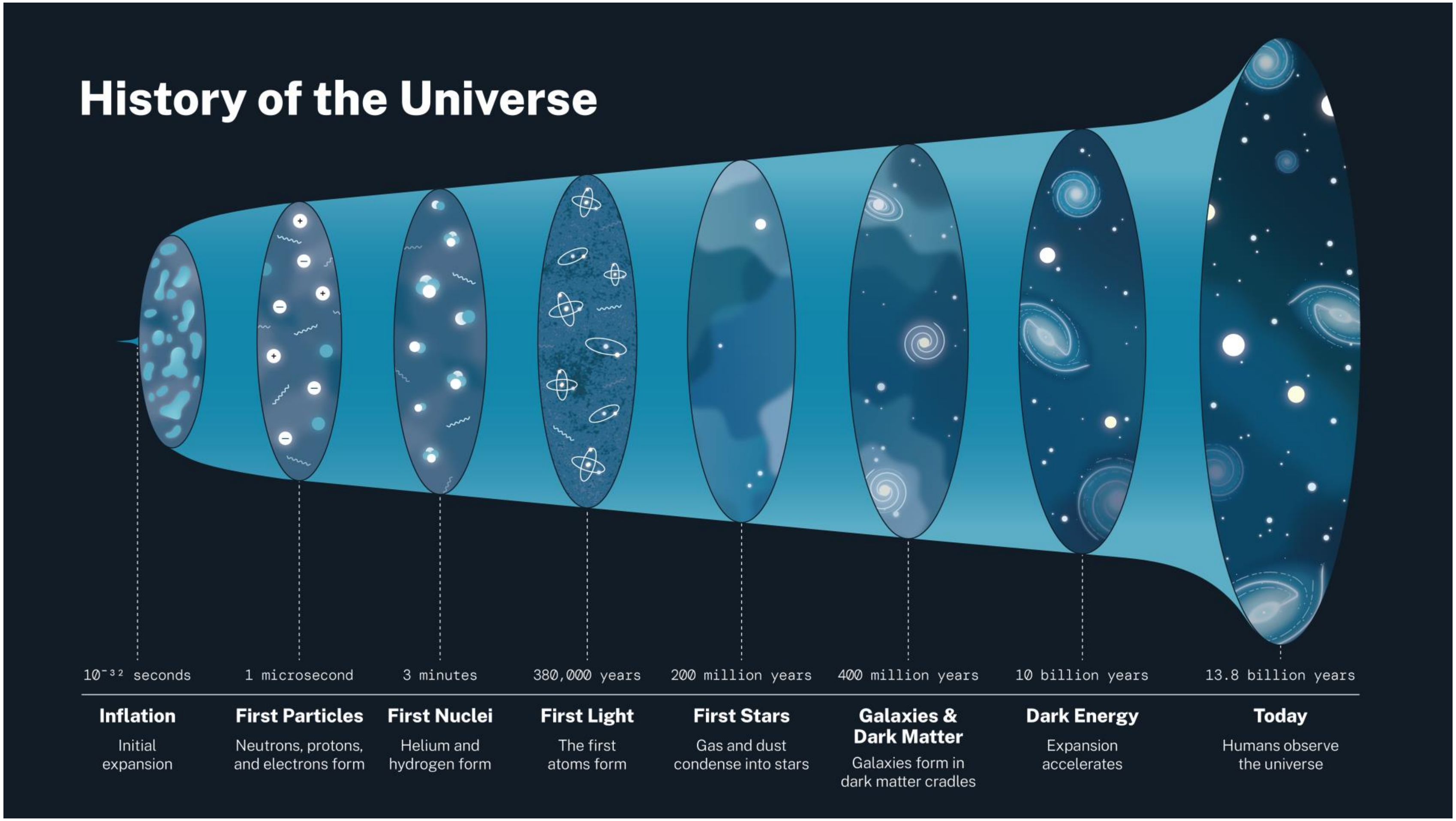
$$\text{where}\ \ \Omega_{k,0} \equiv 1-\Omega_0 = -\frac{k}{H_0^2},\ w_k = -\frac{1}{3}$$

- Quite interesting that *curvature* lies at boundary between DE and non DE, not accelerate nor decelerate the expansion of the Universe.

Universe as a horn!



Stretched horn!

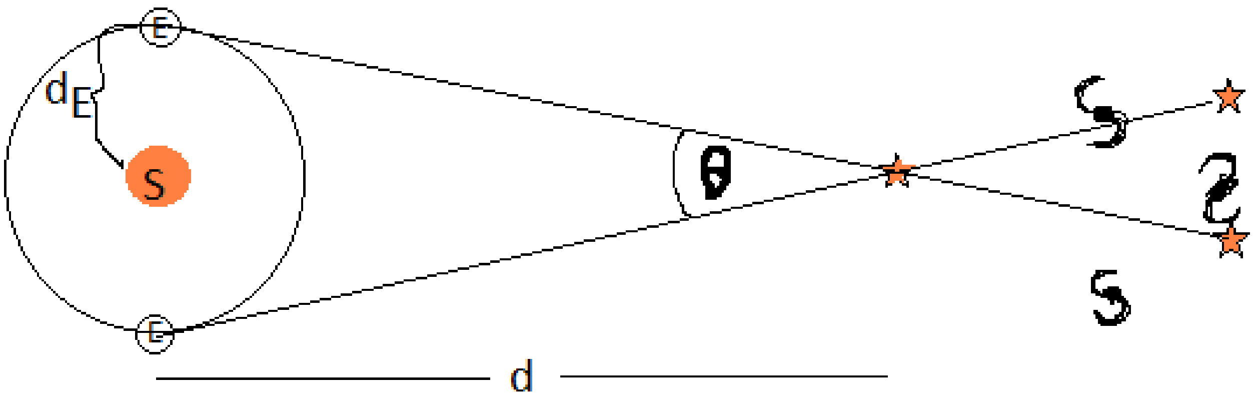


Credit:https://cmb.winterscoming.no/pdfs/baumann.pdf

Event	time t	redshift z	temperature T
Inflation	10^{-34} s (?)	–	–
Baryogenesis	?	?	?
EW phase transition	20 ps	10^{15}	100 GeV
QCD phase transition	20 μs	10^{12}	150 MeV
Dark matter freeze-out	?	?	?
Neutrino decoupling	1 s	6×10^9	1 MeV
Electron-positron annihilation	6 s	2×10^9	500 keV
Big Bang nucleosynthesis	3 min	4×10^8	100 keV
Matter-radiation equality	60 kyr	3400	0.75 eV
Recombination	260–380 kyr	1100–1400	0.26–0.33 eV
Photon decoupling	380 kyr	1000–1200	0.23–0.28 eV
Reionization	100–400 Myr	11–30	2.6–7.0 meV
Dark energy-matter equality	9 Gyr	0.4	0.33 meV
Present	13.8 Gyr	0	0.24 meV

Distances in cosmology

1. Parallax: astrophysically short distances determined by parallax.



$$\frac{\theta}{2} \approx \frac{d_E}{d}, \; d_E = 1 \text{ A.U.} = 1.496 \times 10^8 \text{ km},$$

$$1 \text{ parsec} \equiv d \text{ when } \frac{\theta}{2} = 1 \text{ arcseconds} = 1/3600 \text{ degrees},$$

$$1 \text{ pc} = 3.2616 \text{ lyrs}$$

- Hipparcos satellite (ESA): $\theta \approx 7-9\times 10^{-4}$ arcseconds, $d \geq 100$ pc possible!

Distances in cosmology

apparent luminosity $\ell = \dfrac{L}{4\pi d^2}$, $L =$ absolute luminosity,

apparent magnitude $m - m_0 = -2.5\log_{10}\left(\dfrac{\ell}{\ell_0}\right)$,

$m - M = 5(\log_{10} d - 1)$; d in parsec,

absolute magnitude M defined at 10 pc.

- For $z > 0.1$, cosmological expansion *non-negligible*, how to determine distances at large z ??

1. At time t_0 that light reaches Earth, proper area seen by source is

$$4\pi r_1^2 R^2(t_0),\ r_1 = \text{coord. distance between Earth\&source}$$

Distances in cosmology

2. Arrival rate of photons is lower than emitted rate by $\dfrac{R(t)}{R(t_0)} = \dfrac{1}{1+z}$

3. $E_{\text{received }\gamma} = E_{\text{emitted }\gamma} \dfrac{R(t)}{R(t_0)} = E_{\text{emitted }\gamma} \left(\dfrac{1}{1+z} \right)$

Therefore, $\ell = \dfrac{L}{4\pi r_1^2 R_0^2 (1+z)^2} \equiv \dfrac{L}{4\pi d_L^2},$

$d_L = r_1 R_0 (1+z),$ luminosity distance

- For light travels radially since

$$ds^2 = c^2 dt^2 - R^2(t) \left(\frac{dr^2}{1-kr^2} + r^2 d\Omega \right), k = 1, 0, -1$$

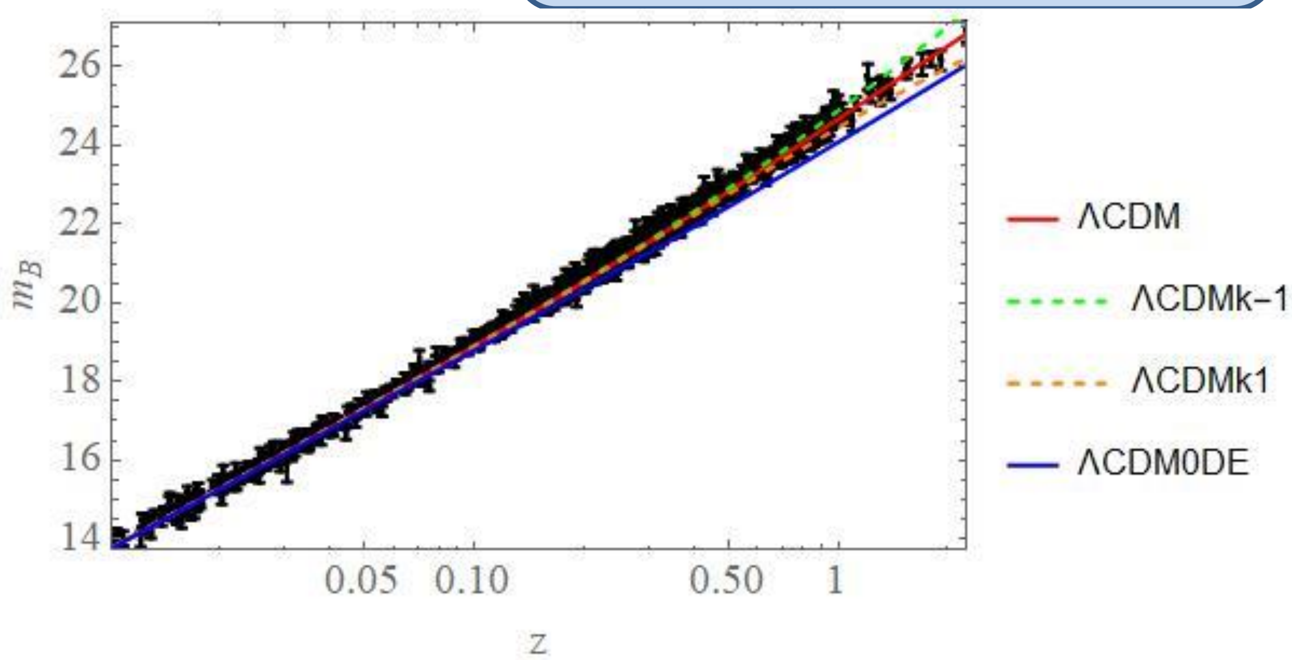
$$\int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}} = \int_0^t \frac{dt}{R(t)} = \int_{R_0}^{R} \frac{dR}{R\dot{R}} = \int_z^0 \frac{1}{H(z)R_0} dz,$$

$$(\text{Arcsin}(r_1), r_1, \text{Arcsinh}(r_1)) = \int_z^0 \frac{1}{H(z)R_0} dz, \text{ for } k = 1, 0, -1$$

Luminosity distance

$$d_L = R_0(1+z)\left[\sin\left(\int_z^0 \frac{dz}{H(z)R_0}\right), \left(\int_z^0 \frac{dz}{H(z)R_0}\right), \sinh\left(\int_z^0 \frac{dz}{H(z)R_0}\right)\right], k = [1, 0, -1]$$

$$m = M + 5\log_{10}\left(\frac{d_L}{\text{pc}} - 1\right)$$



Pantheon, 1048 SNIa data, doi:10.3847/1538-4357/ac8b7a

Universe is accelerating with DE for ***z*<2.3**

- Late time DE dominating era: recall $\frac{R}{R_0} = \frac{1}{1+z}$ so Universe was started to be DE dominated when its roughly half the size of present!
- DE → k → matter → radiation → Inflation(?)

$$\rho \sim R^{>-2} \Rightarrow R^{-2} \Rightarrow R^{-3} \Rightarrow R^{-4} \Rightarrow \rho_{\Lambda}(???)$$

- Lets study Thermal History since Radiation era.

Thermal History

- Natural units: space = time, E = m = 1/space

$$c = 1 \Rightarrow\ 2.998\times10^{8}\ \text{meter} = 1\ \text{second},\ 1\ \text{kg} = 1\ \text{Joule} = \frac{\text{GeV}}{1.602\times10^{-10}},$$

$$\hbar = 1 \Rightarrow\ \frac{6.626\times10^{-34}}{2\pi}\ \text{Joule} = 1\ \text{sec}^{-1} = \left(2.998\times10^{8}\ \text{meter}\right)^{-1},$$

$$1\ \text{fm} = \frac{10^{-15}}{3\times10^{8}}\ \text{sec} = \frac{2\pi}{6.626\times10^{-34}\times3\times10^{23}}\ \text{Joule}^{-1} = \frac{2\pi\times1.602\times10^{-10}}{6.626\times10^{-34}\times3\times10^{23}}\text{GeV}^{-1}$$

$$= 5.067\ \text{GeV}^{-1}\ \Rightarrow\ \boxed{1 = 0.197\ \text{GeV fm}}$$

- Include temperature:

$$k_B = 1 \Rightarrow\ 1.38\times10^{-23}\ \text{Joule} = 1\ \text{Kelvin} \Rightarrow 1\ \text{K} = 1.38\times10^{-23}\ \frac{\text{GeV}}{1.602\times10^{-10}},$$

$$\boxed{1\ \text{K} = 8.61\times10^{-14}\ \text{GeV}}$$

Ex: find Planck mass in natural GeV unit

Thermodynamics in expanding Universe

- Early U is in radiation era with ultra-relativistic particles → massive relativistic gas → massive non-relativistic gas

  as U expands. (radiation era → matter era)
- Assume thermal equilibrium & interactions are taken into account by *Boltzmann* eqn.
- From CMB, its safe to assume thermal equilibrium at single T throughout the U.
- U Expansion → $\lambda = \lambda_0 \frac{R}{R_0} = \frac{\lambda_0}{1+z}, T = T_0(1+z)$

Number density, density, pressure

$$n = \int_0^\infty dn_q = \int \frac{g}{(2\pi)^3} \frac{1}{e^{\frac{E-\mu}{T}} \pm 1} d^3\vec{q} = \int_0^\infty \frac{g}{2\pi^2} \frac{1}{e^{\frac{E-\mu}{T}} \pm 1} q^2 dq$$

$$g = \text{d.o.f.} = (\#\text{spin})(\ \#\text{color})(\#\text{degeneracy})$$

$$\rho = \int_0^\infty E dn_q = \int \frac{g}{(2\pi)^3} \frac{E}{e^{\frac{E-\mu}{T}} \pm 1} d^3\vec{q} = \int_0^\infty \frac{g}{2\pi^2} \frac{E}{e^{\frac{E-\mu}{T}} \pm 1} q^2 dq$$

$$P = \frac{1}{3} \int \frac{q^2}{E} dn_q, \quad E = \sqrt{m^2 + q^2}$$

- 1st law thermodynamics → $s = \frac{\rho + P - \mu n}{T}$; entropy density

- **<u>Ex</u>** photon gas $\mu_\gamma = 0,$ true for all massless particles

$$\rho_\gamma = \int \frac{g}{2\pi^2} \frac{E}{e^{\frac{E-0}{T}} - 1} q^2 dq = \int \frac{2}{2\pi^2} \frac{1}{e^{\frac{q}{T}} - 1} q^3 dq$$

$$= \frac{T^4}{\pi^2} \int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{T^4}{\pi^2} \zeta(4)\Gamma(4) = \frac{\pi^2}{15} T^4; P_\gamma = \frac{1}{3}\rho_\gamma$$

$$s = \frac{4}{3}\frac{\rho_\gamma}{T} = \frac{4}{3}\left(\frac{\pi^2}{15}\right) T^3; n_\gamma = \frac{2\zeta(3)}{\pi^2} T^3$$

$$\int_0^\infty \frac{x^n}{e^x + 1} dx = (1 - 2^{-n})\Gamma(n)\zeta(n); \text{fermion}$$

$$\int_0^\infty \frac{x^n}{e^x - 1} dx = \Gamma(n+1)\zeta(n+1); \text{ boson}$$

- Generically for ultra-relativistic particles in thermal equilibrium;

$$\rho_{\text{rad}} = g_{\text{eff}} \frac{\pi^2}{30} T^4 \equiv \left(\sum_i g_i^b + \frac{7}{8} \sum_i g_i^f \right) \frac{\pi^2}{30} T^4; \text{ for } T \gg m$$

- For ultra-relativistic particles at each *own* equilibrium with $T_i \gg m_i$

$$\rho_{\rm rad} = g_{\rm eff}\frac{\pi^2}{30}T^4 \equiv \left(\sum_i g_i^b\left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_j g_j^f\left(\frac{T_j}{T}\right)^4\right)\frac{\pi^2}{30}T^4;\ \text{for } T_i \gg m_i$$

- Generically relativistic particles ignoring μ_i effects;

$$n = \frac{g}{2\pi^2}\int_0^\infty \frac{p^2}{e^{\sqrt{p^2+m^2}/T}\pm 1}dp = \frac{g}{2\pi^2}T^3\int_0^\infty \frac{y^2}{e^{\sqrt{y^2+x^2}}\pm 1}dy \equiv \frac{g}{2\pi^2}T^3 I_\pm(x);$$

$$\rho = \frac{g}{2\pi^2}\int_0^\infty \frac{p^2\sqrt{p^2+m^2}}{e^{\sqrt{p^2+m^2}/T}\pm 1}dp = \frac{g}{2\pi^2}T^4\int_0^\infty \frac{y^2\sqrt{y^2+x^2}}{e^{\sqrt{y^2+x^2}}\pm 1}dy \equiv \frac{g}{2\pi^2}T^4 J_\pm(x)$$

$$x \equiv \frac{m}{T},\ y \equiv \frac{p}{T}$$

Ex: compute

$n_{\gamma,0}, \rho_{\gamma,0}, \Omega_{\gamma,0}h^2$ for $H_0 = 100h$ km/s/Mpc

- Non-relativistic particles(gas&dust) $T_i \ll m_i$

$$n \simeq g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}; T \ll m$$

$$\rho \simeq nm, P = nT \Rightarrow \text{ ideal gas law!}$$

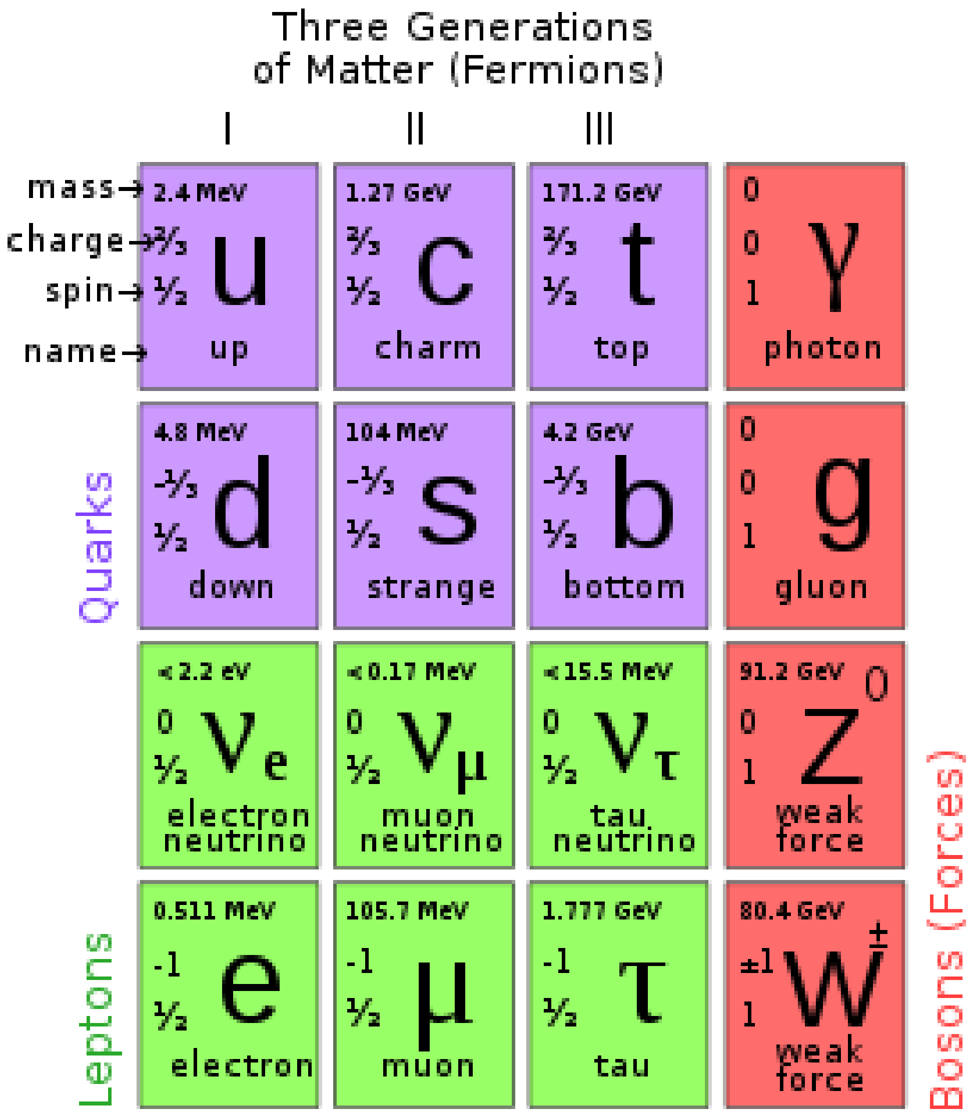
Ex: Show this.

- For neutrinos, there are left-handed 3 flavours & right-handed sterile ??? flavours;

$$\rho_{\nu_L} = \frac{7}{8}\frac{\pi^2}{30}T_\nu^4, \rho_{\nu_R} = \frac{7}{8}\frac{\pi^2}{30}T_{\nu_R}^4 \text{ per flavour}$$

- Has to multiply by 2 to account for particle&antiparticle, neutrinos will decouple the latest due to *small* masses.

Standard Model particles



- +Higgs, m=125 GeV
- Possible to have *sterile neutrinos* which does not interact with anything except via gravity! → warm DM
- Warm DM is harder to reconcile with structure formation.

Evolution of relativistic d.o.f.

$T \geq 100 \text{ GeV} (\simeq 1.2 \times 10^{15} \text{ K}),$

$\exists \, 6 \text{ q}, \bar{\text{q}}, \text{e}^{\pm}, \mu^{\pm}, \tau^{\pm}, \nu_L^{e,\mu,\tau}, \bar{\nu}_L^{e,\mu,\tau}, \text{W}^{\pm}, Z, \gamma, g, H_0;$

$$g_{\text{eff}} = \frac{7}{8}\Big((2\times 12\times 3) + (2\times 6) + (2\times 3)\Big) + \Big((3\times 3) + (2) + (2\times 8) + 1\Big) = 106.75$$

$(\text{q},\bar{\text{q}})$ (charged leptons) $(\nu, \bar{\nu})$; $(\text{W}^{\pm}, Z)$ γ g H_0

$$T \approx 30 \text{ GeV};\ t\bar{t} \rightarrow \gamma\gamma, -(t,\bar{t}); g_{\text{eff}} = 106.75 - \frac{7}{8}(2\times 6) = 96.25$$

$$T \approx 10 \text{ GeV};\ -(\text{W}, \text{Z}, H_0); g_{\text{eff}} = 96.25 - ((3\times 3) + 1) = 86.25$$

$$T < 10 \text{ GeV};\ -(b,\bar{b}); g_{\text{eff}} = 86.25 - \frac{7}{8}(2\times 6) = 75.75$$

$$T > 0.150 \text{ GeV};\ -(c,\bar{c},\tau,\bar{\tau}); g_{\text{eff}} = 75.75 - \frac{7}{8}(12+4) = 61.75$$

Evolution of relativistic d.o.f.

$T \le 0.150$ GeV; confinement;

$\exists n^0, p^+$ (non-rela $\Rightarrow$ small#), $q\bar{q} = \pi^{\pm,0}$ (relativistic), γ, ν, μ, e;

$$g_{\text{eff}} = (2+3) + \frac{7}{8}(6+4+4) = 17.25$$

$T > 1$ MeV; $-(\mu, \pi)$; $g_{\text{eff}} = 17.25 - \frac{7}{8}(4) - 3 = 10.75$

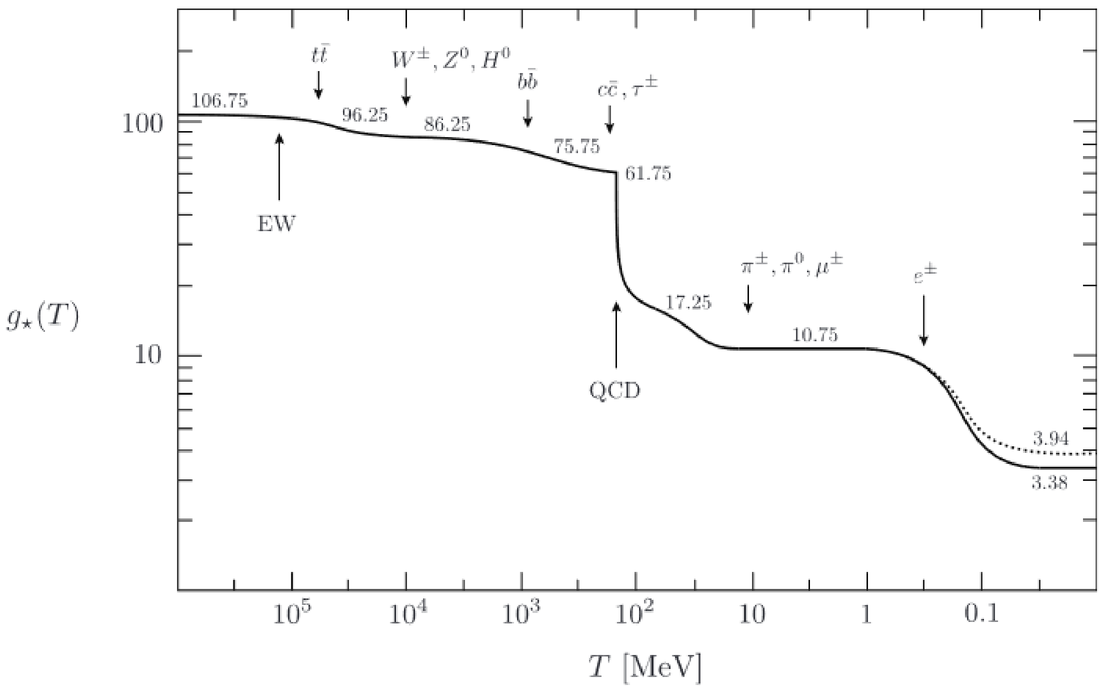


Figure 3.4: *Evolution of relativistic degrees of freedom $g_\star(T)$ assuming the Standard Model particle content. The dotted line stands for the number of effective degrees of freedom in entropy $g_{\star S}(T)$.*

$$g_{\text{eff}} \equiv \left(\sum_i g_i^b \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_j g_j^f \left(\frac{T_j}{T}\right)^4 \right);$$

for $T_i \gg m_i$

Next is $e^+e^- \to \gamma\gamma$ but neutrino will decouple before.

Credit:http://physics.bu.edu/~schmaltz/PY555/baumann_notes.pdf

Entropy conservation in expanding U

- Locally Energy changes in expanding U, but entropy is conserved in thermal equilibrium!
- Non-equilibrium processes produce Entropy.
- *S = S(photons) + S(baryons) + S(DM) + S(BH)*

S(photons) dominates in radiation era.

$$\frac{dS}{dt} = \frac{d}{dt}\left(\frac{\rho + P}{T} V\right) = 0 \ \text{ for } \ \frac{\partial P}{\partial T} = \frac{\rho + P}{T},$$

$$s = \frac{\rho + P}{T} = \sum_i \frac{\rho_i + P_i}{T_i} \equiv \frac{2\pi^2}{45} g_S^{\text{eff}} T^3 = \frac{\text{Const.}}{V} \sim R^{-3}$$

Relativistic d.o.f. of Entropy

$$g_S^{\text{eff}} T^3 R^3 = \text{Const.} \Rightarrow T \sim \left(g_S^{\text{eff}}\right)^{-1/3} \frac{1}{R}$$

$$g_S^{\text{eff}} = \sum_b g_b \left(\frac{T_b}{T}\right)^3 + \frac{7}{8} \sum_f g_f \left(\frac{T_f}{T}\right)^3$$

- Using Friedmann eqn, radiation era;

$$\frac{\dot{R}}{R} = \frac{1}{2t} = \sqrt{\frac{8\pi G}{3} \rho(t)} = T^2 \sqrt{\frac{8\pi G}{3} \frac{\pi^2}{30} g_{\text{eff}}} = \frac{\pi}{3} \sqrt{\frac{g_{\text{eff}}}{10}} \frac{T^2}{M_{Pl}^*},$$

$$M_{Pl}^* \equiv \sqrt{\frac{\hbar c}{8\pi G}} = 2.4 \times 10^{18} \text{ GeV; reduced Planck mass}$$

<u>**Ex**</u>: Show that

$$\frac{T}{1 \text{ MeV}} \simeq 1.5 \left(g_{\text{eff}}\right)^{-1/4} \left(\frac{1 \text{ sec}}{t}\right)^{1/2},$$

$$t \approx 1 \text{ sec} \Rightarrow \text{U has } T \approx 1 \text{ MeV}$$

Neutrino decoupling

- Neutrinos are kept in thermal equilibrium by weak interaction, i.e.,

$$\nu_e + \bar{\nu}_e \leftrightarrow e^- + e^+, e^- + \bar{\nu}_e \leftrightarrow e^- + \bar{\nu}_e$$

- Competing between *scattering* and *U expansion;*

$$t_H = \frac{1}{H}, t_{\text{scattering}} = \frac{1}{\Gamma}, \Gamma = n\bar{v}\sigma = \text{ scattering rate}$$

- Decoupling when scattering time is longer than expansion time, $t_{\text{scattering}} \geq t_H$
- For neutrino with weak interaction,

$$\frac{t_H}{t_{\text{scattering}}} = \frac{\Gamma_{weak}}{H} \approx \left(\frac{T}{\text{MeV}}\right)^3; \; \nu_e, \bar{\nu}_e \text{ decouple around } T \sim \text{MeV}$$

- Neutrinos separate from Standard model particles→

neutrino decoupling.

Two thermal equilibria at T_γ, T_ν

- Same temperature even after decoupling until $e^+e^- \to \gamma\gamma$ at $T \approx m_e c^2 (0.511\text{ MeV})$

$$T_\nu \sim g_{\text{eff}}^S(\text{at } T > m_e c^2) R^{-1},$$

$$T_\gamma \sim g_{\text{eff}}^S(\text{at } T < m_e c^2) R^{-1}$$

- For non-neutrinos;

$$g_{\text{eff}}^S(T > m_e c^2) = 2 + \frac{7}{8}(2 \times 2) = \frac{11}{2},$$

$$g_{\text{eff}}^S(T < m_e c^2) = 2$$

- So after e^+e^- annihilation;

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma \Rightarrow T_{\nu,0} = 1.95\text{ K}$$

- Then,

$$g_{\text{eff}} = 2 + \frac{7}{8}(2 \times N_{\text{eff}})\left(\frac{4}{11}\right)^{4/3} = 3.36,$$

$$g_{\text{eff}}^S = 2 + \frac{7}{8}(2 \times N_{\text{eff}})\left(\frac{4}{11}\right) = 3.94$$

$$n_\nu = \frac{3}{4} N_{\text{eff}} \left(\frac{4}{11}\right) n_\gamma$$

Boltzmann equation

$$\frac{1}{R^3}\frac{d}{dt}(n_1 R^3) = -\alpha n_1 n_2 + \beta n_3 n_4;\ \ \alpha = \langle \sigma v \rangle$$

$\sigma =$ scattering cross section $1+2 \rightarrow 3+4$

At equilibrium, $\frac{d}{dt}(n_1 R^3) = 0 \rightarrow \beta = \left(\frac{n_1 n_2}{n_3 n_4}\right)_{\text{eq}} \alpha$

$$\frac{1}{R^3}\frac{d}{dt}(n_1 R^3) = -\langle \sigma v \rangle \left(n_1 n_2 - \left(\frac{n_1 n_2}{n_3 n_4}\right)_{\text{eq}} n_3 n_4 \right)$$

Another useful form; $N_i \equiv \frac{n_i}{s}, \Gamma_1 = n_2 \langle \sigma v \rangle,$

$$\frac{d \ln N_1}{d \ln R} = -\frac{\Gamma_1}{H}\left(1 - \left(\frac{N_1 N_2}{N_3 N_4}\right)_{\text{eq}} \frac{N_3 N_4}{N_1 N_2} \right)$$

Riccati equation

- If DM is WIMP(Weakly Interacting Massive Particle) and *assuming* $X+\bar{X}\to \ell+\bar{\ell}$
- Then $\frac{dN_X}{dt}=-s\langle\sigma v\rangle\left(N_X^2-(N_X^{\mathrm{eq}})^2\right)\Rightarrow\frac{dN_X}{dx}=-\frac{\lambda}{x^2}\left(N_X^2-(N_X^{\mathrm{eq}})^2\right)$

$$\text{for } x\equiv\frac{M_X}{T},\ \lambda\equiv\frac{\langle\sigma v\rangle M_X^3}{H(M_X)}\frac{2\pi^2}{45}g_S^{\mathrm{eff}}$$

- For $x\ll 1;\ N_X^{\mathrm{eq}}=N_X^{\mathrm{eq}}(x)\approx N_X\approx 1$

  $x\gg 1;\ N_X^{\mathrm{eq}}\approx e^{-x}$

$$N_X^{\infty}\simeq\frac{x_f}{\lambda},x_f\simeq 10$$

Freeze out

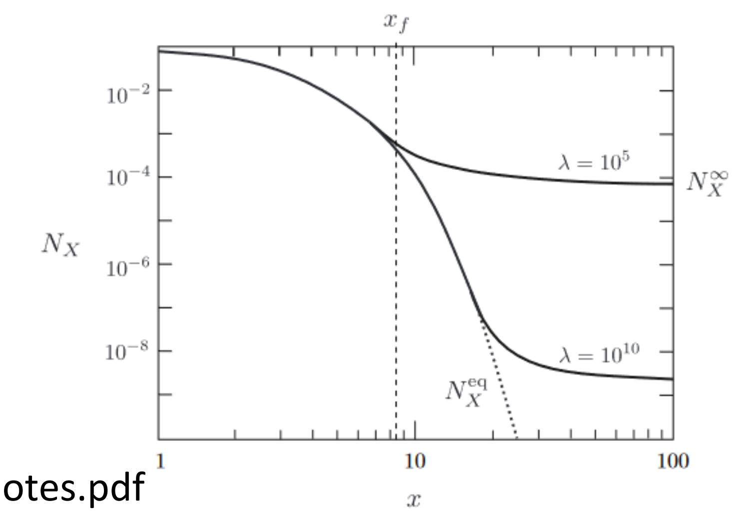


Figure 3.7: *Abundance of dark matter particles as the temperature drops below the mass.*

Credit:http://physics.bu.edu/~schmaltz/PY555/baumann_notes.pdf

WIMP miracle(???)

- From Supernovae fitting and direct counting of visible matter&estimation we estimate matter density 30% of critical density with only 5% baryonic matter; So

$$\Omega_{\mathrm{DM},0} \simeq 0.25 \Rightarrow \Omega_{\mathrm{DM},0} h^2 = 0.11$$

- This could actually relate to WIMP with Weak interaction frozen out during radiation era!

$$\Omega_{X,0} = \frac{\rho_{X,0}}{\rho_{c,0}} = \frac{M_X N_X^\infty s_0}{3M_{Pl}^{*2} H_0^2},\ \rho_{c,0} \equiv \frac{3H_0^2}{8\pi G}, g_{\mathrm{eff}}^S(T_0) = 3.91(\nu\bar{\nu},\gamma)$$

$$\Omega_{X,0} = \frac{M_X x_f s(T_0) H(M_X)}{3M_{Pl}^{*2} H_0^2 \frac{2\pi^2}{45} g_{\mathrm{eff}}^S M_X^3 \langle\sigma v\rangle},\ H(M_X) \simeq \frac{\pi}{3}\left(\frac{g_{\mathrm{eff}}(M_X)}{10}\right)^{1/2} \frac{T_X^2}{M_{Pl}^*}$$

$$\Omega_{X,0} h^2 = \frac{x_f}{\sqrt{10 g_{\mathrm{eff}}(M_X)}} \frac{\pi g_{\mathrm{eff}}^S(T_0)}{9\langle\sigma v\rangle}\left(2.2\times 10^{-10}\,\mathrm{GeV}^{-2}\right) \Rightarrow \langle\sigma v\rangle \simeq 10^{-8}\,\mathrm{GeV}^{-2}$$

Weak scattering!

BUT

- So far at LHC, elsewhere there is no direct evidence of WIMP with weak scattering…
- WTH is DM then???→Please Google or ask Gemini, ChatGPT.
- Actually “gravity” might be modified → modified gravity, or Dark Sector of U, or axion, or sterile neutrino, or other exotic particles
- WTH is DE????

- Next, lets consider *Recombination* where H-atom was formed(coming before Dark Age, First Stars and Reionization).

Credit:https://cmb.wintherscoming.no/pdfs/baumann.pdf

Event	time t	redshift z	temperature T
Inflation	10^{-34} s (?)	–	–
Baryogenesis	?	?	?
EW phase transition	20 ps	10^{15}	100 GeV
QCD phase transition	20 μs	10^{12}	150 MeV
Dark matter freeze-out	?	?	?
Neutrino decoupling	1 s	6×10^9	1 MeV
Electron-positron annihilation	6 s	2×10^9	500 keV
Big Bang nucleosynthesis	3 min	4×10^8	100 keV
Matter-radiation equality	60 kyr	3400	0.75 eV
Recombination	260–380 kyr	1100–1400	0.26–0.33 eV
Photon decoupling	380 kyr	1000–1200	0.23–0.28 eV
Reionization	100–400 Myr	11–30	2.6–7.0 meV
Dark energy-matter equality	9 Gyr	0.4	0.33 meV
Present	13.8 Gyr	0	0.24 meV

Recombination

- H atom is formed and remains when temperature drops below ionization energy of H atom 13.6 eV.

$$X_e = \frac{n_e}{n_b}, \eta = \text{baryon-to-photon ratio}$$

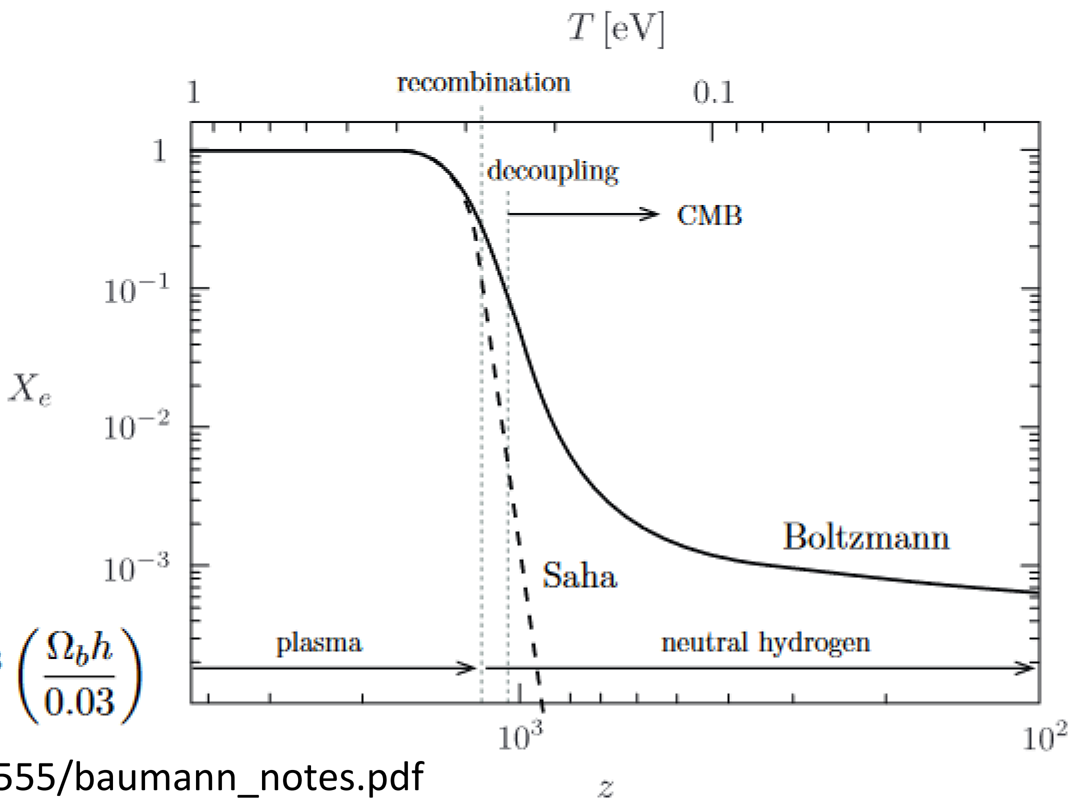
Saha Eqn.

$$\left(\frac{1-X_e}{X_e^2}\right)_{\rm eq} = \frac{2\zeta(3)}{\pi^2}\,\eta\left(\frac{2\pi T}{m_e}\right)^{3/2} e^{B_{\rm H}/T}\,.$$

$$\frac{dX_e}{dx} = -\frac{\lambda}{x^2}\left[X_e^2 - (X_e^{\rm eq})^2\right]$$

$$x \equiv B_{\rm H}/T$$

$$\lambda \equiv \left[\frac{n_b\langle\sigma v\rangle}{xH}\right]_{x=1} = 3.9\times 10^3\left(\frac{\Omega_b h}{0.03}\right)$$



Credit:http://physics.bu.edu/~schmaltz/PY555/baumann_notes.pdf

Figure 3.8: *Free electron fraction as a function of redshift.*

Recombination & Photon decoupling

- Some details:

$$X_e \simeq 0.1 \Rightarrow T_{\rm rec} \simeq 0.3\ {\rm eV} \simeq 3600\ {\rm K},\ z_{\rm rec} \simeq 1320$$

- *Photon decoupling* $e^- + \gamma \leftrightarrow e^- + \gamma\,,$

$$\sigma_T \approx 2 \times 10^{-3}\,{\rm MeV}^{-2}\ \text{is the Thomson cross section}$$

$$\Gamma_\gamma \approx n_e \sigma_T\,,$$

- Reaction ceases when $\Gamma_\gamma(T_{dec}) \sim H(T_{dec})$

$$\Gamma_\gamma(T_{dec}) = n_b X_e(T_{dec})\,\sigma_T = \frac{2\zeta(3)}{\pi^2}\,\eta_b\,\sigma_T\,X_e(T_{dec})T_{dec}^3\,,$$

$$H(T_{dec}) = H_0\sqrt{\Omega_m}\left(\frac{T_{dec}}{T_0}\right)^{3/2}.$$

- Leads to

$$X_{\rm dec} \simeq 0.01 \Rightarrow T_{\rm dec} \simeq 0.27\ {\rm eV},\ z_{\rm rec} \simeq 1100,\ \ t_{\rm CMB} \simeq 380{,}000\ {\rm yrs}$$

BB Nucleosynthesis

- Successful in reproducing $$\frac{n_{\rm n}}{n_p} \simeq \frac{2}{14} \Rightarrow \frac{n_{\rm He}}{n_{\rm H}} \simeq \frac{1}{12} \Rightarrow \frac{m_{\rm He}}{m_H} \simeq \frac{1}{3}$$
- Or He 25%, H 75% by mass, see details elsewhere(e.g. Baumann notes).
- Next, we go back further in order to explain the flatness we see&saw, the validity of cosmological principle in CMB and large-scale homogeneity of matter distribution.
- Inflation is a simple good idea as a quantitative explanation.

Inflation

- *Flatness problem*: why density parameter is so close to 1??? $\Omega(z)-1=\dfrac{\Omega_0-1}{1-\Omega_0+\Omega_{\Lambda,0}R^2+\Omega_{\mathrm{m},0}R^{-1}+\Omega_{\mathrm{r},0}R^{-2}}$

  $\Omega(z)-1\to 0 \text{ as } R\to 0$

- (Use $R^2H^2(1-\Omega)=H_0^2(1-\Omega_0)$ to prove)
- Who tune this at the beginning?→fine tuning problem in cosmology
- Also *Horizon problem*: how CMB equilibrates to 1/100,000 uniformity throughout the entire sky???

Horizons

- Particle horizon = furthest distance we can observe from the PAST.
- Event horizon = furthest distance we can observe in the FUTURE.

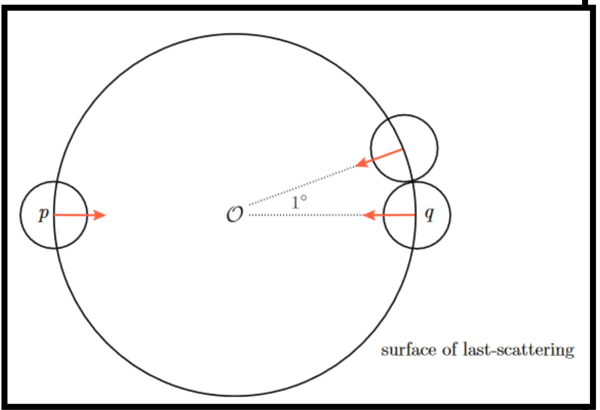
$$ds^2 = c^2dt^2 - R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega\right) \equiv R^2(\eta)(d\eta^2 - d\chi^2)$$

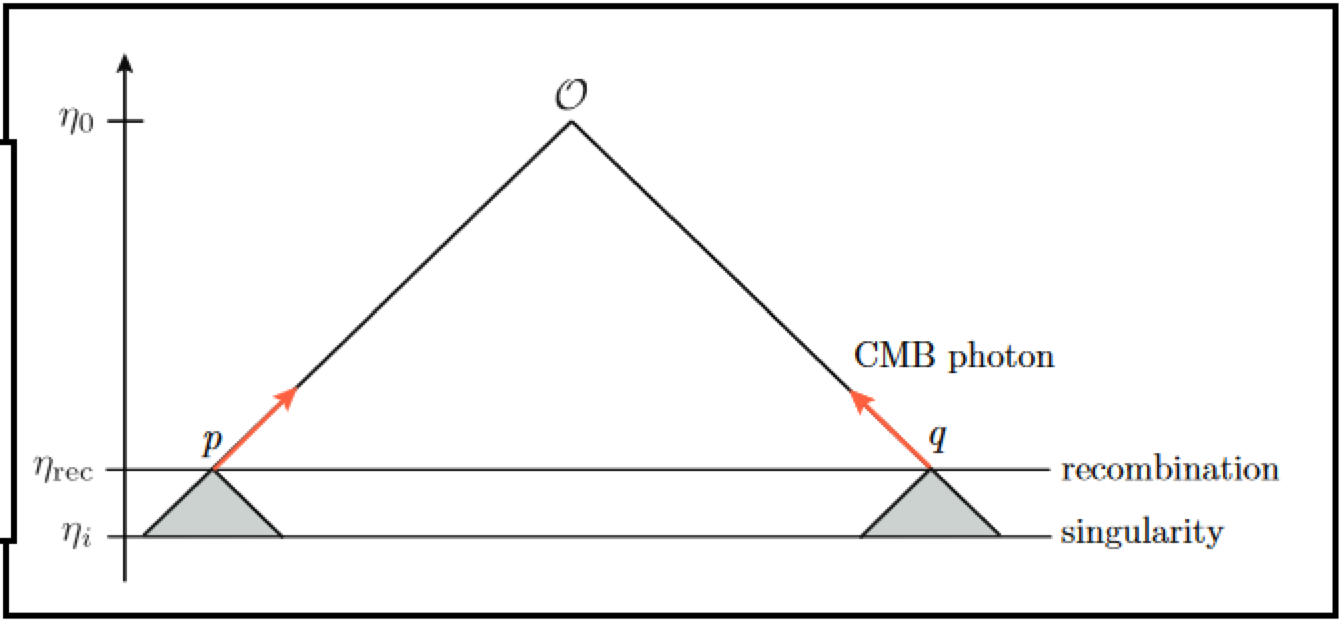
for null path $d\Omega = 0$ in the last step. $\chi = \int \frac{dr}{\sqrt{1-kr^2}}$

- Conformal time η makes things flat and easy to visualize.

$$\chi_{ph} = \eta - \eta_i = \int_{t_i}^{t} \frac{dt}{R(t)}, \chi_{eh} = \eta_f - \eta = \int_{t}^{t_f} \frac{dt}{R(t)}$$

Horizon problem in conformal diagram





Credit:http://physics.bu.edu/~schmaltz/PY555/baumann_notes.pdf

- Past light cones at separate regions cannot be in causal contact, how can they be the same within 1/100,000?
- Introduce concept of *Hubble sphere or Hubble radius*.

$$(RH)^{-1}: \text{comoving Hubble radius}$$

$$\chi_{ph} = \int_{t_i}^{t} \frac{dt}{R(t)} = \int_{\ln R_i}^{\ln R} (RH)^{-1} d\ln R = \eta - \eta_i,$$

$$\text{For } P = w\rho c^2, R(t) = \left(\frac{3}{2} H_0 (1+w) t\right)^{\frac{2}{3(1+w)}}, \chi_{ph} = \left(\frac{2}{1+3w}\right) \frac{1}{H_0} \left(R^{\frac{1+3w}{2}} - R_i^{\frac{1+3w}{2}} \right)$$

$$\text{For } 3w+1>0, \eta_i = \left(\frac{2}{1+3w}\right) \frac{1}{H_0} \left(R_i^{\frac{1+3w}{2}} \right) \to 0 \text{ as } R_i \to 0, \chi_{ph} = \frac{2}{1+3w} (RH)^{-1}$$

$$\chi_{ph} = \frac{2}{1+3w} (RH)^{-1} \text{ for } w = 1/3 \text{ (radiation)}$$

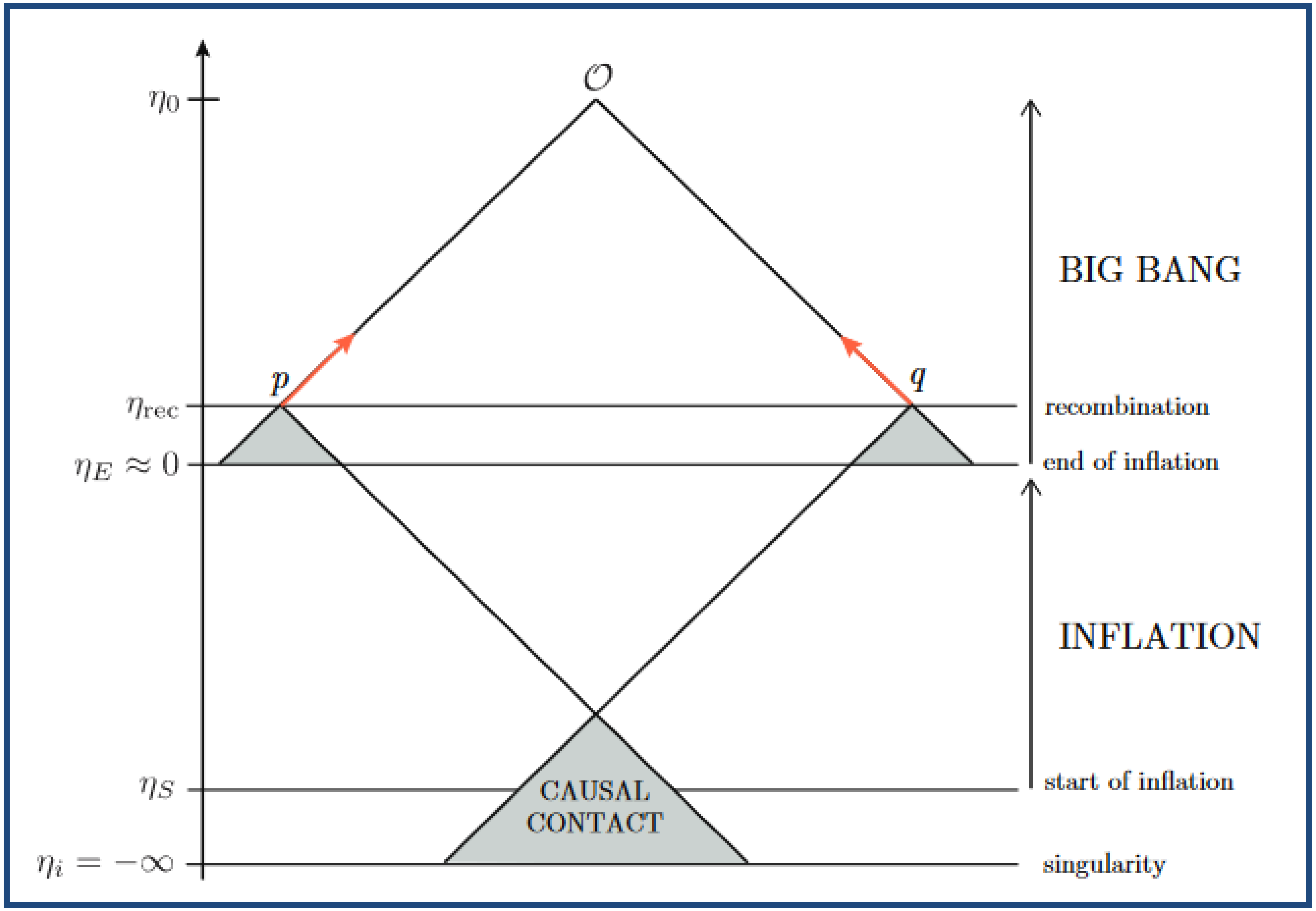
- Hubble radius determines particle horizon.
- Hubble radius can shrink if DE.

$$(RH)^{-1} = H_0^{-1} R^{\frac{1+3w}{2}}, \text{ shrinks if } 1+3w<0$$

$$\text{BUT } \eta_i = \left(\frac{2}{1+3w}\right) \frac{1}{H_0} \left(R_i^{\frac{1+3w}{2}} \right) \to -\infty \text{ as } R_i \to 0 \Rightarrow \chi_{ph} \to \infty!!!$$

- Supernice that particle horizon can be any large!!!

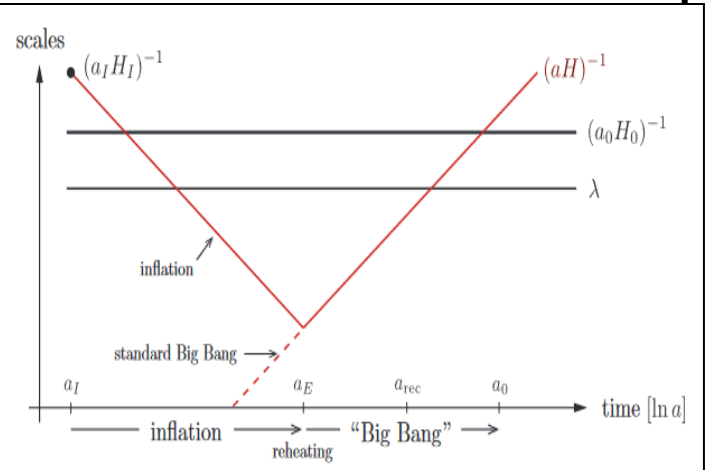
- Extending conformal time to -∞ allows anywhere to be in causal contact in the far past, and the past is infinite to spare with!!!



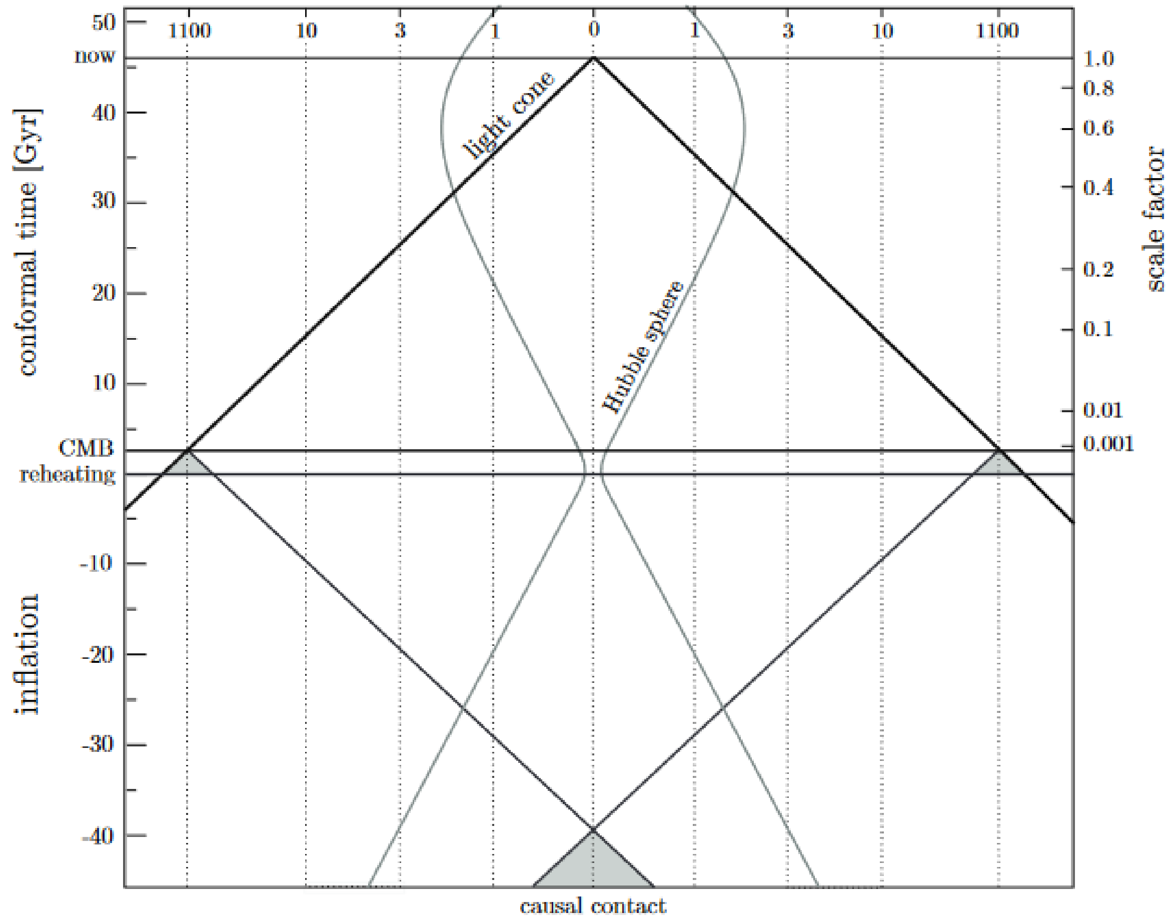
Credit:https://cmb.wintherscoming.no/pdfs/baumann.pdf

- Even before the time of DE in 1998, “DE” was used in Inflation.

https://cmb.wintherscoming.no/pdfs/baumann.pdf



- If we assume GUT as the symmetric point where inflation Ended and Beginning of HOT Big Bang.



To solve horizon problem;

$$(R_I H_I)^{-1} > (R_0 H_0)^{-1} \sim \frac{R_E}{R_0}\left(\frac{H_E}{H_0}\right)(R_E H_E)^{-1} \sim \frac{R_E}{R_0}\left(\frac{R_0}{R_E}\right)^2 (R_E H_E)^{-1}$$

since radiation era $H \sim R^{-2}$, $(R_I H_I)^{-1} > \frac{T_E}{T_0}(R_E H_E)^{-1} = 10^{28}(R_E H_E)^{-1}$

if we set $T_E \sim 10^{15}\,\text{GeV}$, $T_0 \sim 10^{-4}\,\text{eV}$ and $H_I \simeq H_E \Rightarrow \dfrac{R_E}{R_I} > 10^{28}$, e-folding = $28\ln 10 \sim 64$

- Hubble sphere should shrink 10^28 during Inflation until GUT scale era

- Shrinking Hubble sphere ←→ accelerated expansion

$$\frac{d}{dt}(RH)^{-1} = -\left(\frac{\ddot{R}}{\dot{R}^2}\right) < 0 \Leftrightarrow \ddot{R} > 0$$

$$\varepsilon \equiv -\frac{d\ln H}{dN} = -\frac{d\ln H}{d\ln R} = -\frac{\dot{H}}{H^2} = 1 - \frac{\ddot{R}}{\dot{R}^2} < 1 \Leftrightarrow \ddot{R} > 0,$$

$N =$ number of e-folding

- But also has to *last long* enough. Parametrised by

$$\eta \equiv \frac{d\ln\varepsilon}{dN} = \frac{d\ln\varepsilon}{d\ln R} = \frac{\dot{\varepsilon}}{\varepsilon H} \Rightarrow |\eta| < 1 \text{ so that } \varepsilon < 1 \text{ persists.}$$

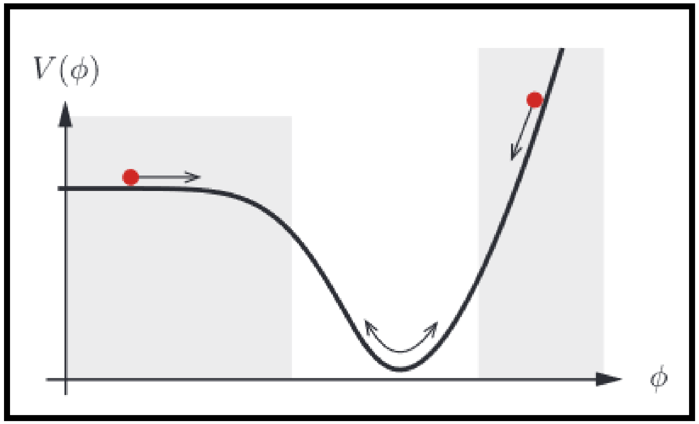
- Lets consider inflation toy model using single scalar field called *inflaton*.

Inflaton toy model

$$Lagrangian = \frac{1}{2}\partial_a\phi\partial^a\phi - V(\phi),\ T_{ab} = \partial_a\phi\partial_b\phi - g_{ab}\left(\frac{g^{cd}}{2}\partial_c\phi\partial_d\phi - V(\phi)\right)$$

$$\rho_\phi = T_t^t = \frac{\dot\phi^2}{2} + V(\phi),\ T_j^i = -P_\phi\delta_j^i = -\delta_j^i\left(\frac{\dot\phi^2}{2} - V(\phi)\right)$$

$$\frac{P_\phi}{\rho_\phi} = \frac{\frac{\dot\phi^2}{2} - V(\phi)}{\frac{\dot\phi^2}{2} + V(\phi)} < -\frac{1}{3} \Leftrightarrow V > \dot\phi^2 \text{ : potential dominates}$$



https://cmb.winterscoming.no/pdfs/baumann.pdf

- Friedmann eqn & conservation eqn lead to:

$$\ddot\phi + 3H\dot\phi + V'(\phi) = 0,\ \text{Klein-Gordon eqn. in FLRW metric}$$

- Roll to oscillate at bottom → *Reheating*

$$\text{For } V \gg \dot\phi^2, \varepsilon \simeq \frac{M_{Pl}^{*2}}{2}\left(\frac{V'}{V}\right)^2 \equiv \varepsilon_V,\ \ \eta_V \equiv \frac{V''}{V}M_{Pl}^{*2};\ \varepsilon_V,\ |\eta_V| \ll 1 \text{ (slow roll)}$$

- Homogeneity&isotropy of CMB requires at least 60 e-foldings until GUT scale 10^15 GeV era if we assume GUT as Beginning of HOT BB.

$$N_{tot} = \int_{R_I}^{R_E} d\ln R = \int_{t_I}^{t_E} H(t)dt; \;\; \varepsilon(t_I) = \varepsilon(t_E) = 1$$

$$N_{tot} = \int_{\phi_I}^{\phi_E} \frac{H}{\dot{\phi}} d\phi = \int_{\phi_I}^{\phi_E} \frac{1}{M^*_{Pl}\sqrt{2\varepsilon}} d\phi \approx \int_{\phi_I}^{\phi_E} \frac{1}{M^*_{Pl}\sqrt{2\varepsilon_V}} d\phi$$

$$N_{CMB} = \int_{\phi_{CMB}}^{\phi_E} \frac{1}{M^*_{Pl}\sqrt{2\varepsilon_V}} d\phi \approx 60 \;\; \Rightarrow N_{tot} > N_{CMB} \text{ required}$$

- Depending on model → constraint on inflaton value at start of inflation, usually superPlanckian inflaton due to required e-folding# >60.
- After end of inflation, need to transfer inflaton Energy to “Big Bang” particles, i.e., particles we see today.
  → This is “HOT Big Bang”.

Reheating

- Energy of inflaton potential needs to transfer to Standard Model particles.

$V(\phi) \approx \frac{m^2}{2}\phi^2$ around minimum, $\ddot{\phi} + 3H\dot{\phi} = -m^2\phi$,

Soon $H \ll m$, ϕ becomes normal oscillatory field with $\langle \phi \rangle, \langle \dot{\phi} \rangle = 0$,

Conservation eqn. $\dot{\rho}_\phi = -3H(\rho_\phi + \frac{\dot{\phi}^2}{2} - \frac{m^2}{2}\phi^2) \approx -3H\rho_\phi$ on average in time.

$\rho_\phi \sim R^{-3}$

- Inflaton decay via coupling with SM matter:

$$\dot{\rho}_\phi + 3H\rho_\phi = -\Gamma_\phi \rho_\phi;\ \Gamma_\phi = \text{ inflaton decay rate}$$

SM particles thermalized at $T_{\text{reheating}} \Rightarrow$ radiation era

Problems

- *Baryogenesis*, why there is much more particles than antiparticles??? CP violation in SM is too small to account for this.
- GUT(Grand Unified Theory) valid? SUSY GUT?
  String? Cyclic Universe??? WHAT???
- Inflation predicts almost *scale-invariant* power spectrum which can be tested with Observations.
- See *Cosmological Perturbations*.