Lecturenote for Cosmology with Computations Workshop 2024 (CosCOM2024)

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Contents

1 Review of GR

1.1 Equivalence Principle

The equivalence principle (EP) is one of the crucial idea leading to argument why do we think the spacetime is curved as well as how to generalize idea from SR to GR. In this lecture note, we will classified EP into two parts: Weak Equivalence Principle (WEP) and Strong Equivalence Principle (SEP).

What is the WEP?
For the WEP, it states that "No experiment in mechanics can distinguish between gravitational field and an accelerating frame of reference" This idea have a root from Galileo demonstration shown in Fig. 1. This infers that "All body fall in the same rate in gravitational field." Now let us do the

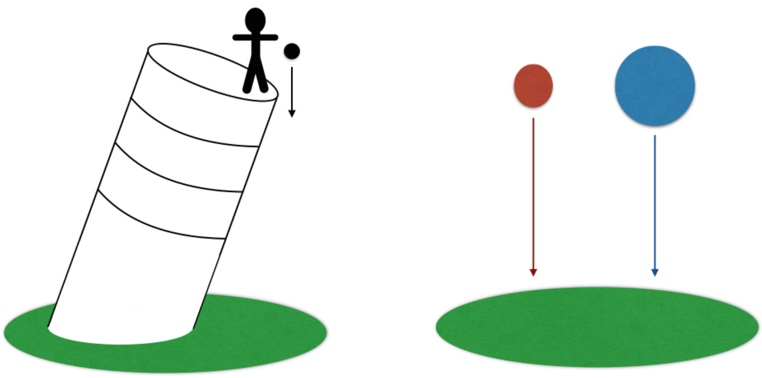


Figure 1: Objects with different masses are dropped to the ground at the same time.

Galileo in a rest box influenced by gravitational field as illustrated in the left panel of Fig. 2. Then changing the situation to a box with an acceleration in the same magnitude of the gravitational field as shown in right panel of Fig. 2. By mean WEP, the results of the experiment are exactly the same. We cannot distinguish both situations by using mechanics experiments.

What is the consequence of WEP?
According to Newton's second law, the particle of mass m_i can be accelerated by applied force as

$$\vec{F} = m_i\vec{a}.$$

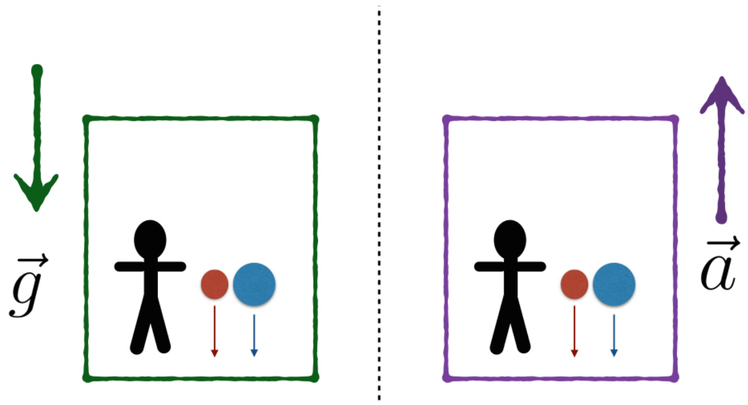


Figure 2: Observers do the experiment in the boxes, then he obtains the same result.

Note that the mass m_i denotes the inertial mass of a particle. This kind of mass play the role to resist of the moment of the particle. The more value of m_i, the more force to make its movement.

There is another kind of mass called "gravitational mass". It measures the response of the particle to the gravitational field,

$$\vec{F} = m_g \vec{E}_g = -m_g \vec{\nabla}\Phi.$$

It plays the similar role to electric charge in terms of electric force, $\vec{F} = q\vec{E}$. Therefore, m_g is essentially gravitational charge. These two kinds of mass are completely different in this sense. By using the argument from the WEP, we then have

$$\vec{F} = m_g \vec{E}_g = m_i \vec{a}, \quad \Rightarrow \quad m_i = m_g.$$

Sometimes, the WEP is known such that "The inertial mass and gravitational mass of any object are equal".

It is important to note that EP is a local principle. To see this property, let us consider the Einstein box, but now it is much bigger as seen in Fig. 3. From this figure, the masses are falling to the center of the Earth, then (for the box large enough) the masses become closer as they falling. In this case, we can distinguish between $\vec{E}_g$ and $\vec{a}$ so that the principle is not valid. Actually, the principle is valid for small enough box. Note that most of gravitational field are not uniform so that the EP treats a gravitational field at a single point which is equivalent to the uniform acceleration.

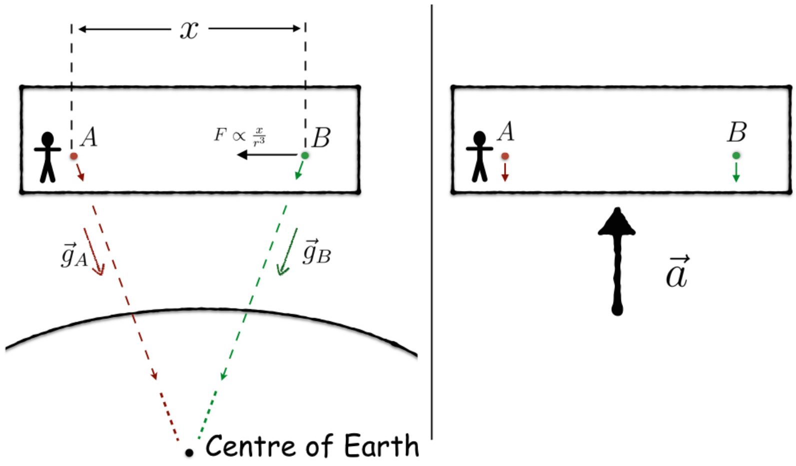


Figure 3: Two masses become closer while they are falling in the large enough box in gravitational field.

Strong Equivalence Principle

For the WEP, it concerns only for the experiments associated with mechanics. It may be ask that Is it possible to use this principle to the other areas of physics? In this consideration we will discuss on the experiments in optics. To see this, let us consider the Einstein box with the laser attached on the top and the receiver in the bottom as illustrated in Fig. 4.By considering two situations, with and without acceleration. From this experiment, one can measure the travel time for both case. By comparing these, we can find the difference of the light wavelength since the speed of light is constant as

$$\frac{\Delta\lambda}{\lambda} = \frac{al}{c^2},$$

where l i the height of the box. This actually is due to the Doppler shift. One can perform the experiment in the same way but now consider the box at rest in gravitational field. It is found that the light wavelength is also shifted in the same way but now it is not the Doppler effect since the box is at rest. We called this effect as gravitational red-shift. From this ons can summarize that "No experiment in optics can distinguish between gravitational field and an accelerating frame of reference". Moreover, we can generalize the idea to all area of physics and this is the statement of the SEP: "All the law of nature are effected in the same way by a gravitational field and a constant acceleration."

More thought-experiment in optics

Set up the experiment in similar way but now let us attach the emitter and

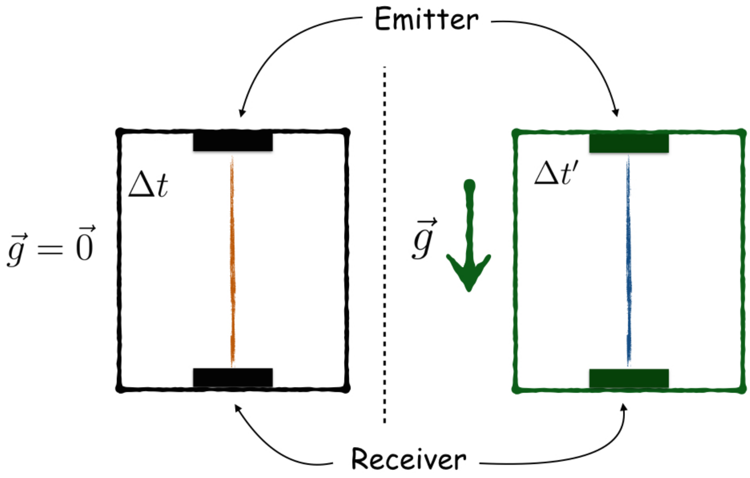


Figure 4: The laser and receiver in the Einstein box with/without gravitational field.

receiver in the horizontal part as shown in Fig. 5. Using EP and this thiught-experinent, one can see that the light travels in a curved path in gravitational field. Note that this conclusion is also obtained from the variational principle: the light follows the path with using minimum time. In any Eucledian space the shortest path is a straingth line, but the curved path in gravitational field. This makes Einstein to obtain the idea that the gravity makes a spacetime curved and then the light moving in such curved spacetime making the curved path.

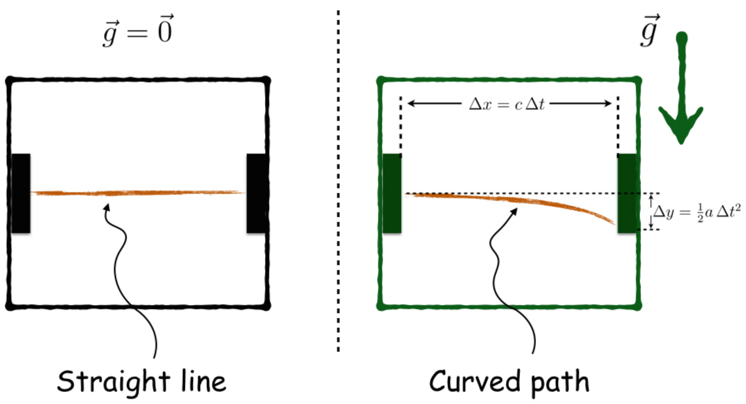


Figure 5: The laser and receiver attaching in horizontal part in the Einstein box with/without gravitational field.

In order to study the gravity, one has to study the curved spacetime. This leads to learn a vector of tensor in curved spacetime and also how to differentiate them. This subject is formally known as ”differential geometry”.

1.2 Spacetime curvature

1.2.1 Intrinsic and Extrinsic curvature

Physically, the curvature can be classified into two types: Intrinsic curvature and extrinsic curvature.

□ Intrinsic curvature: a curvature measured by one in the surface itself do not need the information from the higher dimension.

□ Extrinsic curvature: a curvature measured by one who need the information from the higher dimension.

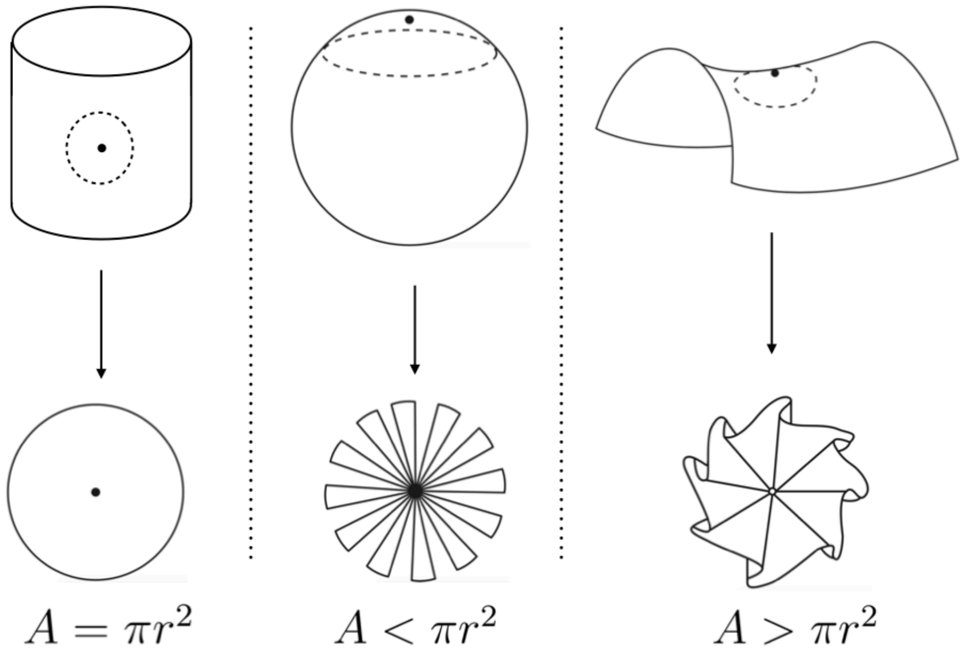


Figure 6: Show how cylinder, sphere and saddle differ in terms of curvature.

Example

1. Cylinder (see the left figure of Fig. 6): In 2 dimensions, it is flat. However, its shape is curved in 3 dimensions. We need the information from 3 dimension in order to identify the curve of cylinder ⇒ Extrinsic curvature. If we unfold the cylinder into 2D, it will look flat paper. In other words, if we write down the circle on the cylinder, it can be put in the flat completely

or the area is still be the same $A = \pi r^2$ as illustrated in the left panel of Fig. 6. In this case, the cylinder is intrinsically flat.

2. Sphere (see the middle figure of Fig6): $\Rightarrow$ From the middle panel, one can see that the circle on the sphere cannot be put in the flat completely or the area is less than the usual one $A < \pi r^2$. This object is intrinsically curved.

3. Saddle (see the right figure of Fig6): $\Rightarrow$ From the right panel, one can see that the circle on the saddle cannot be put in the flat completely or the area is greater than the usual one $A > \pi r^2$. This object is also intrinsically curved.

Let us consider how to defined curvature in mathematical point of view. In order to define the curve properly, one can take a vector in close path. If the vector become exactly the same at the same point, the surface we take such the vector is called flat. If the vector is not the same, the surface is curved.

□ Flat: From Fig. 7, one can see that after we take the vector to the

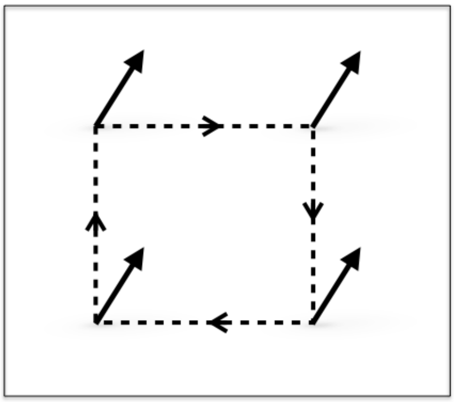


Figure 7: Taking round trip of the vector on flat surface, and then we will found that the vector becomes the same at the same point.

closed path, the vector will be the same. The surface is called flat in this case.

□ Sphere (example of curve): From Fig. 8, We move the vector along $A \to B \to C \to D \to A$. It is found that we do not obtain the same vector $\vec{V}_i = \vec{V}_f \neq \vec{V}_i$. $\to$. It is the curved surface.

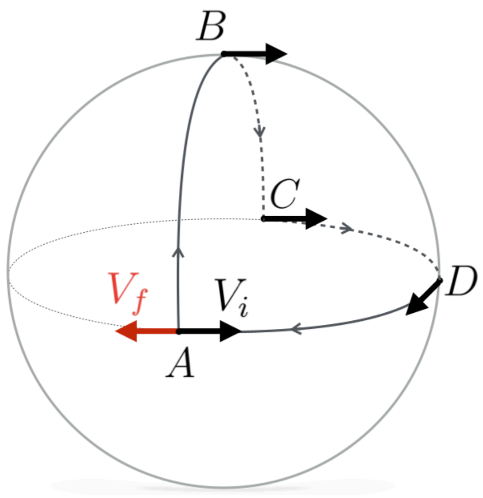


Figure 8: Taking round trip of the vector on flat surface, and then we will found that the vector is not the same.

1.2.2 Vector in curved space

From the previous section, in order to proper deal with spacetime curvature, we have to define the vector on the surface. However, we need to firstly define the mathematical object referred to the curved surface. Mathematically, such an object is called manifold. Conceptually, the manifold is mathematical object which is smooth and locally flat. The locally flat property is compatible to the notion obtained in EP. This allows us to connect the vector defined in flat space to curved manifold. For rigorously speaking, the definition of manifold is any set that can be continuously parameterized (We would not consider more detail about this definition).

Now let us move to consider the vector. Recall the 3-vector, it can be written in terms of basis and components as follows

$$\begin{aligned} \vec{V} &= V_x\hat{i} + V_y\hat{j} + V_k\hat{k}, && (1) \\ &= V_r\vec{e}_r + V_r\theta\vec{e}_\theta + V_\phi\vec{e}_\phi, && (2) \\ &= V^i\vec{e}_i. && (3) \end{aligned}$$

For four dimensional spacetime (in SR), it can be promoted to a vector in n-dimensional manifold as

$$V = V^\mu\vec{e}_\mu, \tag{4}$$

where μ runs over $0, 1, 2, 3$.

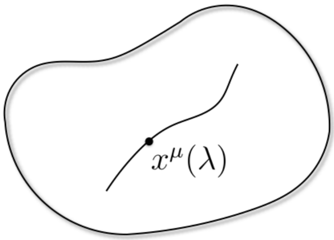


Figure 9: A curve on the manifold $\mathbb{M}$ parametrized by λ with assignment of coordinates $x^\mu(\lambda)$ at a particular point

□ For a manifold $\mathbb{M}$, the curve on the manifold can be parametrized by a parameter λ.
□ A point on the curve can be assigned by the local coordinates $x^\mu(\lambda)$ as shown in Fig. 9.
□ Supposed $f(x^\mu)$ is a function on manifold (Mapping from $\mathbb{M}$ to $\mathbb{R}$), then we can write

$$f\left(x^\mu(\lambda)\right) = g(\lambda) \in \mathbb{R}. \tag{5}$$

The differential of the function along the curve can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} g(\lambda) = \frac{\partial f}{\partial x^\mu} \frac{\partial x^\mu}{\partial \lambda}. \tag{6}$$

As a result, one can write the object in similar form of vector in flat space as

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\partial x^\mu}{\partial \lambda} \frac{\partial}{\partial x^\mu} = \frac{\partial x^\mu}{\partial \lambda} \partial_\mu, \tag{7}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \text{ is a vector,} \qquad \frac{\partial x^\mu}{\partial \lambda} \text{ is a component of vector,} \qquad \partial_\mu \text{ is a basis of vector.} \tag{8}$$

Note that since $\frac{\mathrm{d}}{\mathrm{d}\lambda}$ varies with point in manifold, it plays the role of vector field. Note also that there other non-coordinate basis but we do not consider in this lecture. In this lecture, we restrict our attention on the coordinate basis.
□ **Tangent space**
Vector defined at point p lie in the tangent space denoted by T_p. At point p, there are actually many curves and then the plane tangent to the surface at this point is visualized as T_p as shown in Fig 10. In other words, T_p is obtained by taking all possible curves passing point p. Note that vector

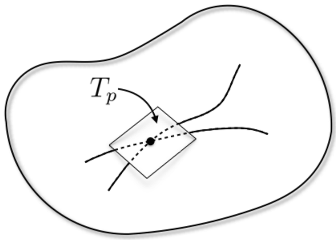


Figure 10: Tangent space

defined at two different points (different T_p) have no relation to each other. This is a crucial property of the vector on manifold which significantly differs from one in flat space.

□ **General coordinate transformation**

In SR, the vector is constructed under the Lorentz transformation. It is valid only in flat spacetime (Minkowski spacetime). This is just a local frame of the curved manifold. For GR, we need to generalize the transformation to cover the whole manifold. In this sense, we introduce the general coordinate transformation (GCT) as

$$x^\mu \to x'^\mu(x). \tag{9}$$

Now let us consider how the component of the vector change under GCT

$$\begin{aligned}\partial_\mu f(x) &= \partial_\mu f(x'(x)),\\ &= \frac{\partial f}{\partial x'^\nu}\frac{\partial x'^\nu}{\partial x^\mu},\\ &= \frac{\partial x'^\nu}{\partial x^\mu}\partial'_\nu f.\end{aligned} \tag{10}$$

Then, we obtain the transformation of the basis as follows

$$\partial_\mu = \frac{\partial x'^\nu}{\partial x^\mu}\partial'_\nu, \qquad \text{or} \qquad \vec{e}_\mu = \frac{\partial x'^\nu}{\partial x^\mu}\vec{e}\,'_\nu, \tag{11}$$

where $\frac{\partial x'^\nu}{\partial x^\mu}$ is GCT matrix. For the case of the matrix is invertible, we can write

$$\partial'_\nu = \frac{\partial x^\mu}{\partial x'^\nu}\partial_\mu, \qquad \text{or} \qquad \vec{e}\,'_\nu = \frac{\partial x^\mu}{\partial x'^\nu}\vec{e}_\mu. \tag{12}$$

with the property

$$\frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\rho}{\partial x'^\nu} = \frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x'^\rho}{\partial x^\nu} = \delta^\mu_\nu \tag{13}$$

As a result, the vector, V under GCT can be rewritten as

$$\begin{aligned}\vec{V} &= V^{\mu}\vec{e}_{\mu}, \\ &= V^{\mu}\left(\frac{\partial x'^{\nu}}{\partial x^{\mu}}\vec{e}'_{\nu}\right),\end{aligned}$$

In order to obtain that the vector is unchanged ($\vec{V} = V^{\mu}\vec{e}_{\mu} = V'^{\nu}\vec{e}'_{\nu} = \vec{V}'$), the component must be transformed as

$$V'^{\nu} = \frac{\partial x'^{\nu}}{\partial x^{\mu}}V^{\mu}. \tag{14}$$

Note that the Lorentz transformation is a special case of GCT.

$$\Lambda^{\mu}{}_{\nu} = \frac{\partial x'^{\nu}}{\partial x^{\mu}}. \tag{15}$$

□ **Dual vector**

There exist other object on manifold called "dual vector". To visualize this kind of vector, let us consider the infinitesimal transformation of coordinate, $x \to x + \mathrm{d}x$. For any function $f = f(x)$ on manifold, it is changed as

$$\mathrm{d}f = \frac{\partial f}{\partial x^{\mu}}\mathrm{d}x^{\mu} \tag{16}$$

By comparing to the usual form of vector, we have

$$\begin{aligned} df &\quad \text{is a dual vector,} \\ \frac{\partial f}{\partial x^{\mu}} &\quad \text{is a component of dual vector,} \\ dx^{\mu} &\quad \text{is a basis of dual vector.}\end{aligned}$$

Mathematically, the dual vector is a map of the vector to $\mathbb{R}$. In the same fashion as vector, $\vec{V}$ defined in T_p, the dual vector $\vec{w}$ is defined in T_p^* (called dual tangent space or cotangent space). By choosing f being the new coordinate, x'^{μ}, we obtain the transformation rule of the coordinate of dual vector as

$$\mathrm{d}x'^{\nu} = \frac{\partial x'^{\nu}}{\partial x^{\mu}}\mathrm{d}x^{\mu}. \tag{17}$$

Similar to a vector, the component of dual vector should be transform

$$w'_{\nu} = \frac{\partial x^{\mu}}{\partial x'^{\nu}}w_{\mu}, \tag{18}$$

in order to obtain the invariant dual vector under GCT, $w = w_\mu \mathrm{d}x^\mu = w'_\mu \mathrm{d}x'^\mu = w'$.

□ **Tensor**

Tensor is an object which combines the description of both dual vector and vector. More precisely, object which has components such that the Cartesian product of "r" basis vector and "s" basis of dual vector. Therefore, rank (r, s) tensor can be defined

$$T = T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s} \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_r} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_s}, \tag{19}$$

The transformation rule can be written as

$$T'^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s} = \frac{\partial x'^\mu_1}{\partial x^\rho_1} \cdots \frac{\partial x'^\mu_r}{\partial x^\rho_r} \frac{\partial x^\sigma_1}{\partial x'^\nu_1} \cdots \frac{\partial x^\sigma_s}{\partial x'^\nu_s} T^{\rho_1 \cdots \rho_r}{}_{\sigma_1 \cdots \sigma_s} \tag{20}$$

Notation Convention

$$T^{\mu_1 \cdots \mu_{r-1}\rho}{}_{\nu_1 \cdots \nu_{s-1}\rho} \quad : \quad (r-1, s-1) \text{ tensor}, \tag{21}$$

$$T^{(\mu_1 \cdots \mu_r)}{}_{\nu_1 \cdots \nu_s} = \frac{1}{r!}(\text{Sum over all permutation}), \tag{22}$$

$$T^{\mu_1 \cdots \mu_r}{}_{[\nu_1 \cdots \nu_s]} = \frac{1}{s!}(\text{Alternative sum over all permutation}). \tag{23}$$

e.g.

$$T^{(\rho\sigma)}{}_\mu = \frac{1}{2}\left(T^{\rho\sigma}{}_\mu + T^{\sigma\rho}{}_\mu\right), \tag{24}$$

$$T^\rho{}_{[\mu\nu]} = \frac{1}{2}\left(T^\rho{}_{\mu\nu} - T^\rho{}_{\nu\mu}\right). \tag{25}$$

It is important to note that, if the exist the metric tensor $g_{\mu\nu}$ on the manifold, it is called Riemanian manifold. From SR, the spacetime is a kind of rigid body and the metric tensor $\eta_{\mu\nu}$ is somehow fixed. However, as we have mentioned, the spacetime is a kind of flexible object in GR view point. Physically, the metric tensor is the object characterized the flexibility of spacetime. Therefore, the curvature of the spacetime will be related to the property of the metric tensor. We will see later on this issue.

Note that we will use $g_{\mu\nu}$ for curved spacetime, while the flat Minkowski spacetime is dented by $\eta_{\mu\nu}$. As a result, the tranformation for the metric tensor can be gneralized as

$$\eta'_{\mu\nu} = \Lambda^\rho_\mu \Lambda^\sigma_\nu \eta_{\rho\sigma} \Rightarrow g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}. \tag{26}$$

The other properties such as dot product of vectors, raise or lower indecies still be the same.

1.2.3 Covariant derivatives

In order to find the proper derivatives on curved spacetime, let consider the transformation of the derivative of V^μ,

$$\begin{aligned}
\partial'_\mu V'^\nu &= \frac{\partial}{\partial x'^\mu} V'^\nu, \\
&= \left(\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial}{\partial x^\rho}\right)\left(\frac{\partial x'^\nu}{\partial x^\sigma}V^\sigma\right), \\
&= \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x'^\nu}{\partial x^\sigma}\frac{\partial}{\partial x^\rho}V^\sigma + \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial^2 x'^\nu}{\partial x^\rho \partial x^\sigma}V^\sigma, \\
&= \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x'^\nu}{\partial x^\sigma}\partial_\rho V^\sigma + \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial^2 x'^\nu}{\partial x^\rho \partial x^\sigma}V^\sigma. \qquad (27)
\end{aligned}$$

We see that $\partial_\mu V^\nu$ does not transform like a (1,1) tensor (because there exists the exceed term, $\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial^2 x'^\nu}{\partial x^\rho \partial x^\sigma}V^\sigma$ in the above equation). This is due to the differentiation of a vector in this manner defined by comparing two vectors in different tangent space. In this sense, the normal derivative can be performed only in the flat spacetime of comparing $V^i(x^j)$ and $V^i(x^j + dx^j) = V^i + dV^i$ (see Fig. 11) as follows

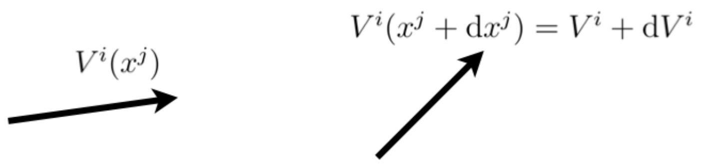


Figure 11: In 3D, two vectors in different points can be compared only in flat space

$$\frac{\partial V^i}{\partial x^j} = \lim_{\mathrm{d}x^j \to 0} \frac{V^i(x^j + \mathrm{d}x^j) - V^i(x^j)}{\mathrm{d}x^j}. \qquad (28)$$

According to the curved spacetime, one has to compare two vectors in the same point (same tangent space). Therefore, we have to move the vector to the same point. Let denote the vector resulting from this transport as $V^\mu(x^\nu) + \delta V^\mu$ as shown in Fig. 12.

Thus the genuine derivative should be constructed from the difference between $V^\mu(x^\nu) + \delta V^\mu$ and $V^\mu(x^\nu) + \mathrm{d}V^\mu$ as

$$\mathrm{D}V^\mu = (V^\mu(x^\nu) + \mathrm{d}V^\mu) - (V^\mu(x^\nu) + \delta V^\mu) = \mathrm{d}V^\mu - \delta V^\mu. \qquad (29)$$

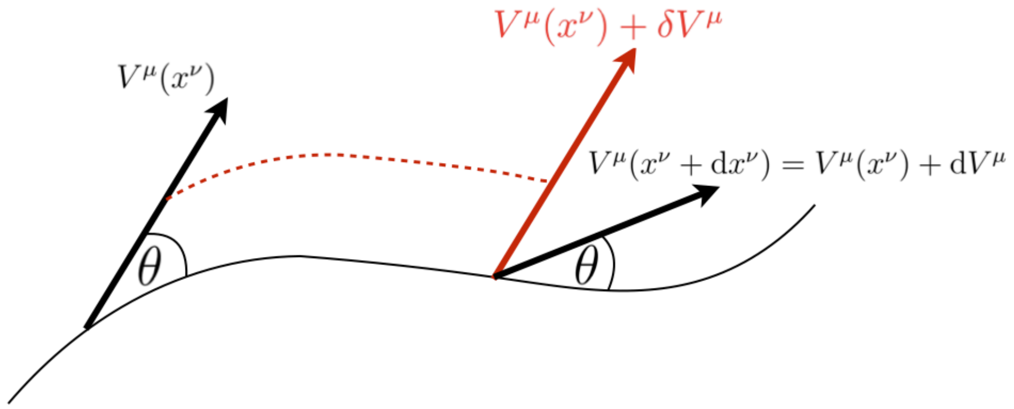


Figure 12: Two vectors in different points can be compared in curved spacetime, if we move one of them to the same point.

Since δV^μ is obtained by parallel transport of vector V^ρ, it should be depend on the vector V^ρ itself. Moreover, it is in the same tangent vector of $\mathrm{d}V^\sigma$ so that it has to proportional to $\mathrm{d}x^\sigma$. As a result we have,

$$\delta V^\mu \propto V^\rho \mathrm{d}x^\sigma \qquad \rightarrow \qquad \delta V^\mu = -\Gamma^\mu_{\rho\sigma} V^\rho \mathrm{d}x^\sigma, \tag{30}$$

where, $\Gamma^\mu_{\rho\sigma}$ is the connection coefficient. Considering derivative (29) along the curve parametrized by a parameter λ, it reads

$$\begin{aligned} \frac{\mathrm{D}V^\mu}{\mathrm{d}\lambda} &= \frac{\mathrm{d}V^\mu}{\mathrm{d}\lambda} - \frac{\delta V^\mu}{\mathrm{d}\lambda}, \\ &= \frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda}\frac{\partial V^\mu}{\partial x^\sigma} + \Gamma^\mu_{\rho\sigma} V^\rho \frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda}, \\ \frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda}\nabla_\sigma V^\mu &= \left(\frac{\partial V^\mu}{\partial x^\sigma} + \Gamma^\mu_{\rho\sigma} V^\rho\right)\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda}. \end{aligned} \tag{31}$$

Then the component of the covariant derivative of vector V^μ can be written as

$$\nabla_\sigma V^\mu = \partial_\sigma V^\mu + \Gamma^\mu_{\rho\sigma} V^\rho. \tag{32}$$

For the dual vector one can find the covariant derivative from the fact that scalar quantity does not change under the transport,

$$\nabla_\sigma (V^\mu W_\mu) = 0. \tag{33}$$

<u>Exercise</u> show that the component of covariant derivative of the dual vector

can be written as

$$\nabla_\sigma W_\mu = \partial_\sigma W_\mu - \Gamma^\rho_{\sigma\mu} W_\rho. \tag{34}$$

Generally, one can find the covariant derivative of the tensor as

$$\begin{aligned} \nabla_\rho T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} &= \partial_\rho T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} + \sum_{i=1}^{p} \Gamma^{\mu_i}_{\rho\sigma} T^{\mu_1\cdots\mu_{i-1}\sigma\mu_{i+1}\cdots\mu_p}{}_{\nu_1\cdots\nu_q} \\ &\quad - \sum_{i=1}^{q} \Gamma^{\sigma}_{\rho\nu_i} T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_{i-1}\sigma\nu_{i+1}\cdots\nu_q}. \end{aligned} \tag{35}$$

<u>Exercise</u> Show that the connection, $\Gamma^\rho_{\mu\nu}$ transforms as the transformation rule

$$\Gamma'^\rho_{\mu\nu} = \frac{\partial x'^\rho}{\partial x^\sigma}\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\Gamma^\sigma_{\alpha\beta} + \frac{\partial x'^\rho}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x'^\mu \partial x'^\nu}. \tag{36}$$

One found that $\Gamma^\rho_{\mu\nu}$ does not transform as tensor. However, $\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}$ transforms as tensor. Therefore it is possible the define the torsion tensor as

$$T^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} \tag{37}$$

Note that the torsion is a resulting from the round trip transport of the scalar function $T^\rho_{\mu\nu} \propto [\nabla_\mu, \nabla_\nu] f$. In GR, we consider only the torsion-free spacetime $T^\rho_{\mu\nu} = 0$. In principle, $\Gamma^\rho_{\mu\nu}$ does not depend on the metric tensor $g_{\mu\nu}$. It is a structure we introduce to the manifold like $g_{\mu\nu}$. However, if we impose the metric compatibility $\nabla_\rho g_{\mu\nu} = 0$, it will depend on the metric tensor can called the Christoffel symbol.

<u>Exercise</u> By using the metric compatibility $\nabla_\rho g_{\mu\nu} = 0$, show that

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right). \tag{38}$$

<u>Exercise</u> Show that

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g} V^\mu\right). \tag{39}$$

1.2.4 Parallel transport and geodesic equation

Considering directional derivative along a vector $\vec{U}$, it reads

$$\left.\frac{\mathrm{D}V^\nu}{d\lambda}\right|_U = U^\mu \nabla_\mu V^\nu. \tag{40}$$

Now let us specify the direction of the derivetives along the tangent vector $(U \to t)$,

$$\left.\frac{\mathrm{D}\ V^\nu}{d\lambda}\right|_t = t^\mu \nabla_\mu V^\nu, \tag{41}$$

$$= \frac{dx^\mu}{d\lambda}\nabla_\mu V^\nu. \tag{42}$$

From this equation, one can see that it imples the transport of the vector a curve parametrized by λ. If we want to keep the vector constant along the path, the covariant derivative of the vector shoould not be changed. In this sense, one can define the parallel transport as

$$\left.\frac{\mathrm{D}V^\nu}{d\lambda}\right|_t = t^\mu \nabla_\mu V^\nu = 0. \tag{43}$$

The properties of parallel transport can be listed as

$$\to \qquad \frac{\mathrm{D}}{\mathrm{d}\lambda} T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} = t^\mu \nabla_\mu T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} = 0. \tag{44}$$

$$\to \qquad \frac{\mathrm{D}}{\mathrm{d}\lambda}(g_{\mu\nu}V^\mu W^\nu) = 0. \tag{45}$$

$$\left(\text{Check : } \cancelto{0}{\frac{\mathrm{D}}{\mathrm{d}\lambda} g_{\mu\nu}} V^\mu W^\nu + g_{\mu\nu}\cancelto{0}{\frac{\mathrm{D}V^\mu}{\mathrm{d}\lambda}} W^\nu + g_{\mu\nu}V^\mu \cancelto{0}{\frac{\mathrm{D}W^\nu}{\mathrm{d}\lambda}} = 0\right).$$

$\to$ If $C(\lambda)$ is an arbitrary curve, the tangent vectors are not parallel-transported into the tangent vectors. (46)

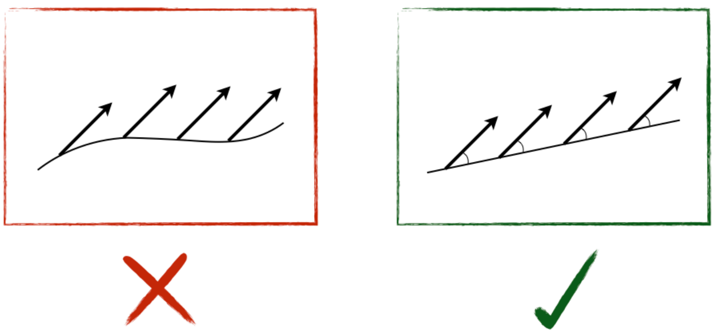


Figure 13: There exists a special subset of all arbitrary curves to satisfy the tangent vectors are parallel-transported into the tangent vectors

There exists a special subset of all arbitrary curves to satisfy the tangent vectors are parallel-transported into the tangent vectors. This curve is called "geodesic path" or auto-parallel curve as shown in Fig 13. From the condition for parallel transport,

$$\begin{aligned} t^\mu \nabla_\mu t^\rho &= 0, \\ \left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\right)\left(\partial_\mu t^\rho + \Gamma^\rho_{\mu\nu} t_\nu\right) &= 0, \\ \frac{\mathrm{d}t^\rho}{\mathrm{d}\lambda} + \Gamma^\rho_{\mu\nu} t_\mu t_\nu &= 0, \\ \frac{\mathrm{d}^2 x^\rho}{\mathrm{d}\lambda^2} + \Gamma^\rho_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} &= 0. \end{aligned} \tag{47}$$

This is called geodesic equation. The parameter, λ satisfying the geodesic equation is called affine parameter. In the other word,

Parallel transport of V^μ along geodesic path.
$\Downarrow$
Moving vector by fixing direction and magnitude.

It is important to note that the geodesic equation can be derived from the variational principle. Generally, it can be written as

$$\ddot{x}^\rho + \Gamma^\rho_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = f(\lambda)\dot{x}^\rho, \tag{48}$$

From this equation, it infers that the geodesic equation can be obtained from the notion of the parallel transport by fixing only the direction of the vector V^μ, (not fixing the magnitude). However, this is still equivalent since we can find other proper affine parameter λ.
Exercise Show that, it is possible to find the other affine parameter to obtain the usual geodesic equation.

1.2.5 Curvature tensor

As we have mentioned before, in order to find the intrinsic curvature, we have to move the vector in closed path. In this sense, it is natural to use the parallel transport to move the vector. Therefore, in this subsection, we will find the spacetime curvature by performing the parallel transport of a vector along the closed path as shown in Fig. 14.

It is found that, in general, the change of vector is

$$\delta V^\rho = -\Gamma^\rho_{\mu\sigma} V^\sigma \mathrm{d}x^\mu. \tag{49}$$

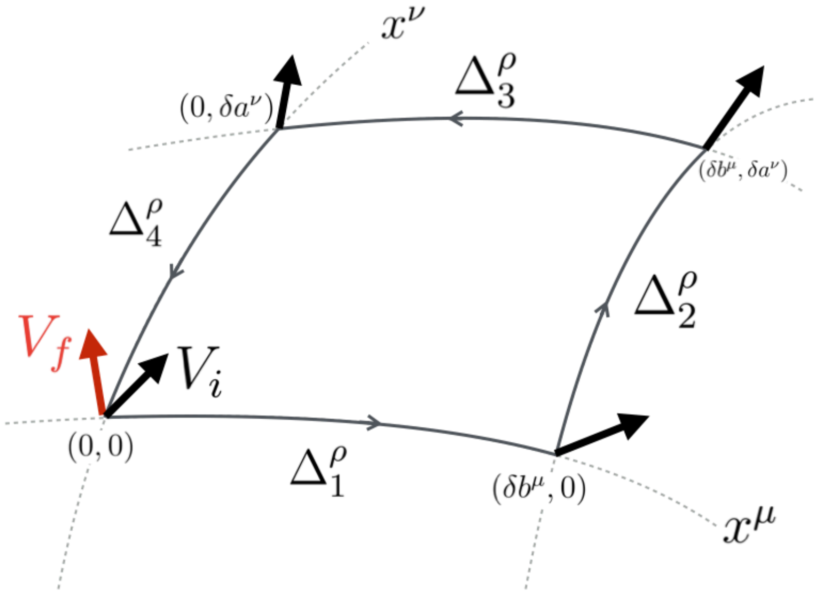


Figure 14: Round trip parallel transport of a vector

According to the above figure, we have

$$\Delta_1^\rho = V_i^\rho(B) - V_i^\rho(A) = -\int_{x^\nu=0} \Gamma^\rho_{\mu\sigma} V^\sigma \mathrm{d}x^\mu, \tag{50}$$

$$\Delta_2^\rho = V_i^\rho(C) - V_i^\rho(B) = -\int_{x^\nu=\delta b^\nu} \Gamma^\rho_{\mu\sigma} V^\sigma \mathrm{d}x^\mu, \tag{51}$$

$$\Delta_3^\rho = V_i^\rho(D) - V_i^\rho(C) = \int_{x^\mu=\delta a^\mu} \Gamma^\rho_{\mu\sigma} V^\sigma \mathrm{d}x^\mu, \tag{52}$$

$$\Delta_4^\rho = V_i^\rho(A) - V_i^\rho(D) = \int_{x^\mu=0} \Gamma^\rho_{\mu\sigma} V^\sigma \mathrm{d}x^\mu. \tag{53}$$

Then

$$\begin{aligned}
\Delta_1^\rho + \Delta_3^\rho &= \left(\int_{x^\nu=\delta a^\nu} - \int_{x^\nu=0}\right) \Gamma^\rho_{\mu\sigma} V^\sigma \mathrm{d}x^\mu, \\
&\approx \delta a^\nu \int \partial_\nu \left(\Gamma^\rho_{\mu\sigma} V^\sigma\right) \mathrm{d}x^\mu, \\
&\approx \delta a^\nu \delta b^\mu \partial_\nu \left(\Gamma^\rho_{\mu\sigma} V^\sigma\right). \qquad (54)
\end{aligned}$$

and

$$\begin{aligned}
\Delta_2^\rho + \Delta_4^\rho &= \left(-\int_{x^\mu=\delta b^\mu} + \int_{x^\mu=0}\right)\Gamma^\rho_{\mu\sigma}V^\sigma \mathrm{d}x^\mu, \\
&\approx -\delta b^\mu \int \partial_\mu\left(\Gamma^\rho_{\nu\sigma}V^\sigma\right)\mathrm{d}x^\nu, \\
&\approx -\delta b^\mu \delta a^\nu \partial_\mu\left(\Gamma^\rho_{\nu\sigma}V^\sigma\right). \qquad (55)
\end{aligned}$$

The whole parallel transportation can be written as

$$\begin{aligned}
\Delta^\rho &= \sum_i \delta_i^\rho, \\
&\approx \delta a^\nu \delta b^\mu \left[\left(\partial_\nu \Gamma^\rho_{\mu\sigma} - \partial_\mu \Gamma^\rho_{\nu\sigma}\right)V^\sigma + \Gamma^\rho_{\nu\sigma}\partial_\mu V^\sigma - \Gamma^\rho_{\mu\sigma}\partial_\nu V^\sigma\right], \\
&= \delta a^\nu \delta b^\mu \left[\left(\partial_\nu \Gamma^\rho_{\mu\sigma} - \partial_\mu \Gamma^\rho_{\nu\sigma}\right)V^\sigma + \Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\mu\lambda}V^\lambda - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda}V^\lambda\right], \\
&= \delta a^\nu \delta b^\mu \left(\partial_\nu \Gamma^\rho_{\mu\sigma} - \partial_\mu \Gamma^\rho_{\nu\sigma} + \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}\right)V^\sigma, \\
&= \delta a^\nu \delta b^\mu R^\rho{}_{\sigma\nu\mu}V^\sigma. \qquad (56)
\end{aligned}$$

where the tensor $R^\rho{}_{\sigma\nu\mu} \equiv \partial_\nu \Gamma^\rho_{\mu\sigma} - \partial_\mu \Gamma^\rho_{\nu\sigma} + \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}$ called Riemannian tensor which describes how the manifold curves.

Next we will discuss the alternative approach to define the Riemannian tensor. Let us consider the parallel transportation of vector along different 2 paths in which the starting and ending points are the same as seen in Fig. 15.

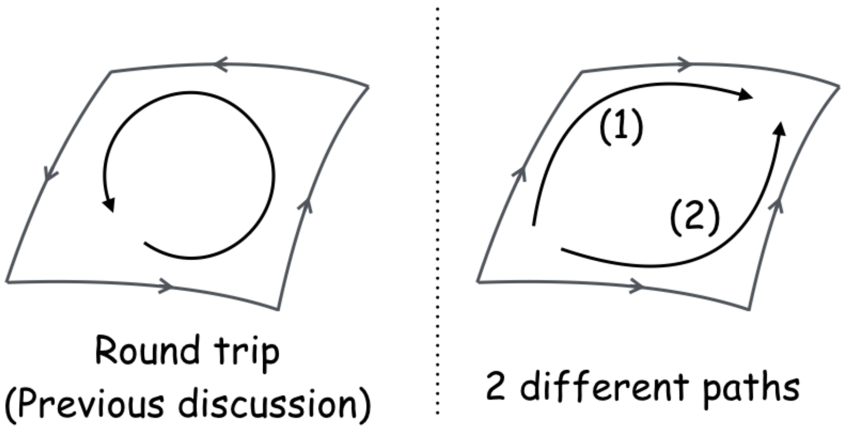


Figure 15: Other way to obtain the curvature tensor

The difference of parallel transported vectors along path 1 and path 2 can be written as

$$
\begin{aligned}
\left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) V^\rho &= \partial_\mu\left(\nabla_\nu V^\rho\right) - \Gamma^\sigma_{\mu\nu}\nabla_\sigma V^\rho + \Gamma^\rho_{\mu\sigma}\nabla_\nu V^\sigma - \partial_\nu\left(\nabla_\mu V^\rho\right) + \Gamma^\sigma_{\nu\mu}\nabla_\sigma V^\rho - \Gamma^\rho_{\nu\sigma}\nabla_\mu V^\sigma \\
&= \partial_\mu\left(\cancel{\partial_\nu V^\rho} + \Gamma^\rho_{\nu\gamma}V^\gamma\right) - \partial_\nu\left(\cancel{\partial_\mu V^\rho} + \Gamma^\rho_{\mu\gamma}V^\gamma\right) \\
&\quad + \Gamma^\rho_{\mu\sigma}\left(\partial_\nu V^\sigma + \Gamma^\sigma_{\nu\gamma}V^\gamma\right) - \Gamma^\rho_{\nu\sigma}\left(\partial_\mu V^\sigma + \Gamma^\sigma_{\mu\gamma}V^\gamma\right) - \left(\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu}\right)\nabla_\sigma V^\rho, \\
&= \partial_\mu\Gamma^\rho_{\nu\gamma}V^\gamma + \cancel{\Gamma^\rho_{\nu\gamma}\partial_\mu V^\gamma} - \partial_\nu\Gamma^\rho_{\mu\gamma}V^\gamma - \cancel{\Gamma^\rho_{\mu\gamma}\partial_\nu V^\gamma} \\
&\quad + \Gamma^\rho_{\mu\sigma}\left(\cancel{\partial_\nu V^\sigma} + \Gamma^\sigma_{\nu\gamma}V^\gamma\right) - \Gamma^\rho_{\nu\sigma}\left(\cancel{\partial_\mu V^\sigma} + \Gamma^\sigma_{\mu\gamma}V^\gamma\right) - \left(\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu}\right)\nabla_\sigma V^\rho, \\
&= \left(\partial_\mu\Gamma^\rho_{\nu\gamma} - \partial_\nu\Gamma^\rho_{\mu\gamma} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\gamma} - \Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\mu\gamma}\right)V^\gamma - 2\Gamma^\sigma_{[\mu\nu]}\nabla_\sigma V^\rho, \\
&= R^\rho{}_{\gamma\mu\nu}V^\gamma - 2\Gamma^\sigma_{[\mu\nu]}\nabla_\sigma V^\rho.
\end{aligned}
$$

Notice that this difference between 2 vectors is considered in the spacetime with torsion because the term $\Gamma^\sigma_{[\mu\nu]}$ does not vanish. Hence, for torsionless spacetime, we also have

$$
\left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right)\left(V^\rho W_\rho\right)
\begin{cases}
= 0, & \text{for torsionless spacetime,} \\
\neq 0, & \text{for spacetime with torsion.}
\end{cases}
\tag{58}
$$

Next, we will consider in the torsionless spacetime. Let us first define

$$
O_{\mu\nu} \equiv \nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu. \tag{59}
$$

From (1.2.5), we can write

$$
\begin{aligned}
O_{\mu\nu}V^\rho W_\rho + V^\rho O_{\mu\nu}W_\rho &= 0, \\
V^\rho O_{\mu\nu}W_\rho &= -R^\rho{}_{\gamma\mu\nu}V^\gamma W_\rho.
\end{aligned}
\tag{60}
$$

The difference of dual vector parallel transported along paths gives us

$$
O_{\mu\nu}W_\rho = -R^\gamma{}_{\rho\mu\nu}W_\gamma. \tag{61}
$$

In general, for any (p,q) tensor, we have

$$
\begin{aligned}
O_{\mu\nu}T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} &= R^{\mu_1}{}_{\gamma\mu\nu}T^{\gamma\mu_2\cdots\mu_p}{}_{\nu_1\cdots\nu_q} + \cdots + R^{\mu_i}{}_{\gamma\mu\nu}T^{\mu_1\cdots\mu_{i-1}\gamma\mu_{i+1}\cdots\mu_p}{}_{\nu_1\cdots\nu_q} + \cdots + R^{\mu_p}{}_{\gamma\mu\nu}T^{\mu_1\cdots\mu_{p-1}\gamma}{}_{\nu_1\cdots\nu_q} \\
&\quad - R^\gamma{}_{\nu_1\mu\nu}T^{\mu_1\cdots\mu_p}{}_{\gamma\nu_2\cdots\nu_q} - \cdots - R^\gamma{}_{\nu_i\mu\nu}T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_{i-1}\gamma\nu_{i+1}\nu_q} - \cdots - R^\gamma{}_{\nu_q\mu\nu}T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_{q-1}\gamma}.
\end{aligned}
\tag{62}
$$

1.2.6 Properties of Riemann tensor

From the definition of Riemanian tensor,

$$
R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}. \tag{63}
$$

Contracting with $g_{\gamma\rho}$,

$$\begin{aligned} g_{\gamma\rho}R^{\rho}{}_{\sigma\mu\nu} &= R_{\gamma\sigma\mu\nu} = g_{\gamma\rho}\partial_\mu\Gamma^\rho_{\nu\sigma} - g_{\gamma\rho}\partial_\nu\Gamma^\rho_{\mu\sigma} + g_{\gamma\rho}\Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - g_{\gamma\rho}\Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}, \\ &= \partial_\mu(g_{\gamma\rho}\Gamma^\rho_{\nu\sigma}) - \partial_\mu g_{\gamma\rho}\Gamma^\rho_{\nu\sigma} - \partial_\nu(g_{\gamma\rho}\Gamma^\rho_{\mu\sigma}) + \partial_\nu g_{\gamma\rho}\Gamma^\rho_{\mu\sigma} + g_{\gamma\rho}\Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - g_{\gamma\rho}\Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}, \end{aligned} \tag{64}$$

Choosing to consider in geodesic coordinate ($\Gamma^\rho_{\mu\nu} = 0$, but $\partial_\sigma\Gamma^\rho_{\mu\nu} \neq 0$), we can write

$$\begin{aligned} R_{\gamma\sigma\mu\nu} &= \partial_\mu(g_{\gamma\rho}\Gamma^\rho_{\nu\sigma}) - \partial_\nu(g_{\gamma\rho}\Gamma^\rho_{\mu\sigma}), \\ &= \frac{1}{2}\partial_\mu\left[g_{\gamma\rho}g^{\rho\lambda}\left(\cancel{\partial_\nu g_{\sigma\lambda}} + \partial_\sigma g_{\nu\lambda} - \partial_\lambda g_{\nu\sigma}\right)\right] - \frac{1}{2}\partial_\nu\left[g_{\gamma\rho}g^{\rho\lambda}\left(\cancel{\partial_\mu g_{\sigma\lambda}} + \partial_\sigma g_{\mu\lambda} - \partial_\lambda g_{\mu\sigma}\right)\right] \\ &= \frac{1}{2}\left(\partial_\mu\partial_\sigma g_{\gamma\nu} - \partial_\mu\partial_\gamma g_{\sigma\nu} + \partial_\nu\partial_\gamma g_{\sigma\mu} - \partial_\nu\partial_\sigma g_{\gamma\mu}\right). \end{aligned} \tag{65}$$

It is easily to see that

$$R_{\gamma\sigma\mu\nu} = -R_{\gamma\sigma\nu\mu}, \tag{66}$$
$$R_{\gamma\sigma\mu\nu} = -R_{\sigma\gamma\mu\nu}, \tag{67}$$
$$R_{\gamma\sigma\mu\nu} = R_{\mu\nu\gamma\sigma}. \tag{68}$$

These properties are not only exist in the geodesic coordinates, but there are also in all coordinates.

Let consider

$$\vec{d}\vec{w}_1 = \nabla_{[\nu}a_{\rho]}\,\vec{\theta^\nu}\wedge\vec{\theta^\rho}, \tag{69}$$
$$\vec{d}^2\vec{w}_1 = \nabla_{[\mu}\nabla_\nu a_{\rho]}\,\vec{\theta^\mu}\wedge\vec{\theta^\nu}\wedge\vec{\theta^\rho} = 0. \tag{70}$$

Since the basis does not vanish, the components are

$$\begin{aligned} 0 &= \nabla_{[\mu}\nabla_\nu a_{\rho]}, \\ &= \frac{1}{2}\left(\nabla_{[\mu}\nabla_\nu a_{\rho]} - \nabla_{[\nu}\nabla_\mu a_{\rho]}\right), \\ &= \frac{1}{2}R^\gamma{}_{[\rho\mu\nu]}a_\gamma. \end{aligned} \tag{71}$$

We obtain another properties of Rimannian tensor

$$R^\gamma{}_{[\sigma\mu\nu]} = 0. \tag{72}$$

It can be alternatively written as

$$R_{[\gamma\sigma\mu\nu]} = 0. \tag{73}$$

Eventually, we have 4 independent properties which are (66)-(68) and (72)(or (73)).

Counting degrees of freedom
In n-dimensional spacetime, from the property in (68), we can consider the Riemannian tensor as the 2-rank tensor,

$$R_{\gamma\sigma\mu\nu} = T_{ab}, \tag{74}$$

where the new indices are defined as the pair of the old ones, $a \equiv \gamma\sigma$ and $b \equiv \mu\nu$. Suppose that each a and b contains m degrees of freedom. Since T_{ab} is symmetric, the number of degrees of freedom is $m(m+1)/2$.

From the properties in (66) and (67), it is anti-symmetric for each pair $\gamma\sigma$ and $\mu\nu$. Thus a and b contain $n(n-1)/2$ degrees of freedom. Then the properties (66)-(68) give us that

$$\# \text{ of d.o.f of } R_{\gamma\sigma\mu\nu}\Big|_{\text{using (66)}-\text{(68)}} = \frac{\left(\frac{n(n-1)}{2}\right)\left(\frac{n(n-1)}{2}+1\right)}{2}. \tag{75}$$

However, there is 1 property left which is (72) or (73). We choose to use (73). It is found that (73) will eliminate $n(n-1)(n-2)(n-3)/4!$ degrees of freedom. Thus we have

$$\begin{aligned}
\# \text{ of d.o.f of } R_{\gamma\sigma\mu\nu} &= \frac{\left(\frac{n(n-1)}{2}\right)\left(\frac{n(n-1)}{2}+1\right)}{2} - \frac{n(n-1)(n-2)(n-3)}{4!}, \\
&= \frac{n(n-1)}{24}\left[3n(n-1)+6-(n-2)(n-3)\right], \\
&= \frac{n(n-1)}{24}(2n^2+2n), \\
&= \frac{1}{12}n^2(n-1)(n+1).
\end{aligned} \tag{76}$$

We can see that there are 20 degrees of freedom for 4-dimensional spacetime. Notice that the number of degrees of freedom in 1 dimension is zero. This means that there is no curvature in 1 dimension. We need information in higher dimension in order to identify the line in 1 dimension is straight or curved.

Bianchi Identity
Considering

$$O_{[\mu\nu}\nabla_{\rho]}W_\sigma = -\overset{=0 \text{ by (72)}}{\cancel{R^\lambda{}_{[\rho\mu\nu]}}}\nabla_\lambda W_\sigma - R^\lambda{}_{\sigma[\mu\nu}\nabla_{\rho]}W_\lambda \tag{77}$$

and

$$
\begin{aligned}
\nabla_{[\rho}O_{\mu\nu]}W_\sigma &= -\nabla_{[\rho}\left(R^\lambda{}_{|\sigma|\mu\nu]}W_\lambda\right), \\
&= -\nabla_{[\rho}R^\lambda{}_{|\sigma|\mu\nu]}W_\lambda - R^\lambda{}_{\sigma[\mu\nu}\nabla_{\rho]}W_\lambda. \qquad (78)
\end{aligned}
$$

In order to obtain (77) = (78), the term $\nabla_{[\rho}R^\lambda{}_{|\sigma|\mu\nu]}W_\lambda$ must be zero. Then

$$
\nabla_{[\rho}R_{|\lambda\sigma|\mu\nu]}W^\lambda = 0, \qquad (79)
$$

or

$$
\begin{aligned}
\nabla_{[\rho}R_{|\lambda\sigma|\mu\nu]} &= 0, \\
\nabla_{[\rho}R_{\mu\nu]\lambda\sigma} &= 0, \qquad (80)
\end{aligned}
$$

which is called Bianchi identity.

Ricci tensor
In order to $R^\gamma{}_{\gamma\mu\nu} = 0$
Ricci tensor

$$
R^\gamma{}_{\mu\gamma\nu} = R_{\mu\nu} \qquad (81)
$$

Ricci scalar
Ricci scalar

$$
R = g^{\mu\nu}R_{\mu\nu}. \qquad (82)
$$

Einstein tensor
Contracting (80) with $g^{\rho\lambda}g^{\mu\sigma}$,

$$
\begin{aligned}
g^{\rho\lambda}g^{\mu\sigma}\nabla_{[\rho}R_{\mu\nu]\lambda\sigma} &= g^{\rho\lambda}g^{\mu\sigma}\frac{1}{3}\left(\nabla_\rho R_{\mu\nu\lambda\sigma} + \nabla_\mu R_{\nu\rho\lambda\sigma} + \nabla_\nu R_{\rho\mu\lambda\sigma}\right), \\
0 &= \frac{1}{3}\left[\nabla^\lambda(-R_{\nu\lambda}) + \nabla^\sigma(-R_{\nu\sigma}) + \nabla_\nu R\right], \\
&= -\frac{2}{3}\nabla^\mu\left(R_{\nu\mu} - \frac{1}{2}g_{\nu\mu}R\right), \\
&= -\frac{2}{3}\nabla^\mu G_{\nu\mu}. \qquad (83)
\end{aligned}
$$

The tensor $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is called the Einstein tensor. Note that this tensor is divergence-less, $\nabla^\mu G_{\mu\nu} = 0$ corresponding to the behaviour of matter which obeys the law of conservation of energy. Hence the Einstein tensor is useful to explain the curvature of spacetime due to the existence of matter as we will discuss later.

1.3 Energy momentum tensor

1.3.1 Individual particle

A 4-momentum P^μ can be used to provide a complete description of energy and momentum of a particle, so that the dynamics of the particle can be described. From SR, the 4-momentum P^μ can be written in terms of 4-velocity V^μas

$$P^\mu = mV^\mu, \tag{84}$$

where m is a rest mass of the particle. The 4-velocity can be defined as

$$V^\mu = \frac{dx^\mu}{d\tau}, \tag{85}$$

where τ is the proper time and the 4-velocity obeys the normalization

$$V_\mu V^\mu = -1. \tag{86}$$

The 4-velocity can be thought as a tangent vector along the timelike worldline. Note that, the rest frame of the particle, the 4-velocity can be written as $V^\mu = (1, 0, 0, 0)$.

1.3.2 System of particles

Rather than specify the individual 4-momentum of all particles, we instead describe the system by a ”fluid”. To specify the fluid, one may need to know the the macroscopic quantities such as density, pressure, entropy, viscosity and so no. Therefore, the single 4-momentum of the fluid is not sufficient to describe the fluid. We can go further to describe the fluid by using the symmetric (2,0) tensor called ”Energy Momentum Tensor” (EMT), $T^{\mu\nu}$. A general definition of EMT is the flux of 4-momentum, P^μ across a surface of constant x^ν. In order to explore the physical meaning of each components of energy momentum tensor , let us consider the infinitesimal element of the fluid in its rest frame with a volume V.

- T^{00}: flux of p^0 (energy) in x^0 (time) $\rightarrow$ ”energy density”.
- $T^{0i} = T^{i0}$: flux of p^i (momentum) in x^0 (time) $\rightarrow$ ”momentum density”.
- $T^{ij}(i = j)$: flux of p^i (momentum) in x^j $\rightarrow$ This represents the transfer momentum of element in i direction into j direction corresponding to the force per unit volume in i direction acting on the plane with $x^j =$ constant. Therefore, for $i = j$, this corresponds to the ”pressure”
- $T^{ij}(i \neq j)$: As the same strategy, for $i \neq j$, this corresponds to the ”shear” due to viscosity of the fluid.

1.3.3 Simple and useful example (dust)

Dust is a system of particles which are at rest with respect to each other. The 4-velocity of the fluid is the same as all particles. The number-flux 4-vector can be defined as

$$N^\mu = nV^\mu, \tag{87}$$

where, n is number density of particles measured in their rest frame. Suppose that the particle s have the same mass m, the energy density at the rest frame can be written as

$$\rho_0 = mn, \tag{88}$$

In the rest frame, one can write N^μ and p^μ as $N^\mu = (n, 0, 0, 0)$ and $p^\mu = (m, 0, 0, 0)$, so that the ENT of dust can be written as

$$T^{\mu\nu}_{\text{(dust)}} = p^\mu N^\nu = mnU^\mu U^\nu = \rho_0 U^\mu U^\nu. \tag{89}$$

1.3.4 Conservation of EMT (dust)

To see clearly how EMT conserves, let us consider Minkowski spacetime $g_{\mu\nu} = \eta_{\mu\nu}$. As a result, the conservation equation can be written as

$$\partial_\mu T^{\mu\nu} = 0. \tag{90}$$

From exercise, the EMT in moving frame can be written as $T^{00} = \rho$, $T^{0i} = \rho v^i$, $T^{ij} = \rho v^i v^j$, where $\rho = \rho_0\gamma^2$ and $\gamma = (1 - v^2)^{-1/2}$.

- zero component:

$$\begin{aligned} \partial_\mu T^{\mu 0} &= \partial_0 T^{00} + \partial_i T^{i0}, \\ &= \partial_t \rho + \partial_i(\rho v^i), \\ &= \partial_t \rho + \vec{\nabla}(\rho\vec{v}) = 0. \end{aligned} \tag{91}$$

  This corresponds to the conservation of energy/mass. To see more clearly, let consider the familiar one which is the moving charge with charge density ρ. The total charge and the current density can be written respectively as

$$Q = \int \rho dV, \tag{92}$$

$$\vec{J} = \rho\vec{v}. \tag{93}$$

During time δt, the change of the charges enclosed by the surface A can be written as

$$\delta Q_1 = \frac{\partial Q}{\partial t}\delta t = \int \left(\frac{\partial \rho}{\partial t} dV\right) \delta t. \quad (94)$$

The charges escaping through the surface can be written as

$$\delta Q_2 = \left(\oint \vec{J}\cdot d\vec{a}\right)\delta t = \left(\int \vec{\nabla}\cdot\vec{J} dV\right)\delta t, \quad (95)$$

where we have used divergence theorem. The charges must be conserved, $\delta Q_1 + \delta Q_2 = 0$. As a result, we have

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} &= 0, \\ \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) &= 0. \end{aligned} \quad (96)$$

Comparing to our case, it implies that the equation belongs to the conservation of mass/energy.

- i component:

$$\begin{aligned} \partial_\mu T^{\mu i} &= \partial_0 T^{0i} + \partial_j T^{ji}, \\ &= \partial_t(\rho v^i) + \partial_j(\rho v^j v^i) = 0. \end{aligned} \quad (97)$$

This equation is also in the same form as the previous one so that it corresponds to the conservation equation as well.

As a result, one find that $\partial_\mu T^{\mu\nu} = 0$ is the conservation equation of EMT in flat Minkowski. One can generalize this equation to one in the curved spacetime by replacing ∂_μ by ∇_μ

$$\nabla_\mu T^{\mu\nu} = 0. \quad (98)$$

1.3.5 Newtonian limit

One of important conditions to construct the Einstein equation is that the equation must be reduced to Newtonian theory. Such the conditions are following: 1) A particle must move slowly comparable to the speed of light. 2) the gravitational field should be weak. This condition allows us to use the perturbation method to perform calculation, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu} \ll 1$. Another condition is the gravitational field must be static. This condition is imposed since it provides easy way to compare to the Newtonian theory.

In Newtonian theory for the central force, the gravitational field can be written as

$$\vec{E}_g = -\vec{\nabla}\Phi, \quad \Phi = -\frac{GM}{r}, \tag{99}$$

where Φ is the gravitational potential.

Now let us consider equation in Einstein theory. As we have known, the equation that explain how particle moves due to the curvature of the spacetime is the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0, \tag{100}$$

where $\Gamma^\mu_{\nu\rho}$ are components of affine connection. From the first condition, "moving slowly", one can use the approximation as follows

$$t \approx \tau, \quad \frac{dx^0}{d\tau} \approx 1, \quad \frac{dx^i}{d\tau} \ll 1 \approx 0. \tag{101}$$

Applying this condition to the geodesic equation, one has

$$\begin{aligned} \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} &= 0, \\ \frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{00}\frac{dx^0}{dt}\frac{dx^0}{dt} &= 0, \\ \frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{00} &= 0. \end{aligned} \tag{102}$$

Let us consider the connection by applying the second condition, "weak field limit" and also the static one, $\partial_0 g_{\mu\nu} = 0$, one obtains

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2}g^{\mu\nu}\left(\partial_\rho g_{\nu\sigma} + \partial_\sigma g_{\nu\rho} - \partial_\nu g_{\rho\sigma}\right), \tag{103}$$

$$\begin{aligned} \Gamma^\mu_{00} &= \frac{1}{2}g^{\mu\nu}\left(\partial_0 g_{\nu 0} + \partial_\sigma g_{\nu 0} - \partial_\nu g_{00}\right), \\ \Gamma^\mu_{00} &= -\frac{1}{2}\eta^{\mu\nu}\partial_\nu h_{00}, \\ \Gamma^i_{00} &= -\frac{1}{2}\eta^{ij}\partial_j h_{00}. \end{aligned} \tag{104}$$

Substituting this connection into Eq. (102), the geodesic equation in Newtonian limit can be written as

$$\begin{aligned}
\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{00} &= 0, \\
\frac{d^2x^i}{dt^2} - \frac{1}{2}\delta^{ij}\partial_j h_{00} &= 0, \\
\frac{d^2x^i}{dt^2} - \frac{1}{2}\partial_i h_{00} &= 0, \\
\vec{E}_g &= \frac{1}{2}\vec{\nabla} h_{00}.
\end{aligned} \tag{105}$$

By comparing the gravitational field in Eq. (105) from Einstein theory and one in Eq. (99) from Newtonian theory, one obtains

$$h_{00} = -2\Phi = \frac{2GM}{r} \;\rightarrow\; g_{00} = -\left(1 - \frac{2GM}{r}\right). \tag{106}$$

1.4 Einstein equation

In order to construct the equation for describing the relation between matter/energy and spacetime curvature, one has impose two requirements such that

- The equation must be reduced to Newtonian theory.
- The curvature part must contain the metric tensor $g_{\mu\nu}$ and its derivatives such as $\Gamma^\rho_{\mu\nu}, R^\rho{}_{\mu\sigma\nu}, R_{\mu\nu}$ and R while the matter/enegy should be proportional to the EMT $T_{\mu\nu}$.

From the first requirement, the important equation in Newtonian theory in the Poisson equation $\nabla^2\Phi = 4\pi G\rho$. As we discussed before, the component of the metric tensor is proportional to the gravitational potential $g_{00} \propto \Phi$. Therefore, in order to reduce the master equation into the Poisson equation, the curvature part must be proportional to the second derivative of the metric. These quantities are $R^\rho{}_{\mu\sigma\nu}, R_{\mu\nu}$ and R.

1.4.1 Vacuum equation

For the vacuum equation, the matter/energy part vanishes and then the curvature part will be vanished

$$f(R^\rho{}_{\mu\sigma\nu}, R_{\mu\nu}, R) = 0. \tag{107}$$

One may firstly guess for this equation such that

$$R^\rho{}_{\mu\sigma\nu} = 0\,(?). \tag{108}$$

However, one found that this may not be possible since this equation provides the flat spacetime nearby the massive source. So that we can make a further guess by

$$R_{\mu\nu} = 0\,(?). \tag{109}$$

This is a good choice since the Ricci tensor has ten dof. like the metric tensor, hoping that ten dof. of the metric transfer to ten dof. of Ricci tensor through their second derivatives making from source nearby.

1.4.2 Equation with source

Now let us consider the equation with source. By adding the EMT, the equation may be written as

$$R_{\mu\nu} = kT_{\mu\nu}\,(?), \tag{110}$$

where k is the proportional constant. This still be the good choice since the index symmetry also satisfy. However, as we discussed before, the EMT obeys the conservation equation $\nabla_\mu T^{\mu\nu} = 0$ while the Ricci tensor does not satisfy in general. As a result, one may find other quantities to satisfy this condition as well as maintain the mentioned requirements. Fortunately, from the Bianchi indentity, $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$, it serve us the conservation quantity as follows

$$\nabla^\mu G_{\mu\nu} = 0, \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \tag{111}$$

where $G_{\mu\nu}$ is called Einstein tensor. Note that the derivation of $\nabla^\mu G_{\mu\nu} = 0$. As a result, the equation can be constructed as

$$G_{\mu\nu} = kT_{\mu\nu}. \tag{112}$$

Next task for this construction is that we have to find the proportional constant k as well as check whether this equation satisfies the vacuum equation or not. To perform this evaluation, let us take the trace of the above equation as follows

$$\begin{aligned} R - \frac{1}{2}(4)R &= kT, \\ R &= -kT \end{aligned} \tag{113}$$

Substituting R from this equation into Eq. (112), one obtains

$$\begin{aligned} R_{\mu\nu} + \frac{1}{2} kT g_{\mu\nu} &= kT_{\mu\nu}, \\ R_{\mu\nu} &= k\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right). \end{aligned} \tag{114}$$

From this equation, one can see that the vacuum equation still be satisfied where the source is eliminated, $T_{\mu\nu} = T = 0$.

Now, we will find the proportional constant by taking the Newtonian limit into Eq. (114). As a result, the EMT can be written as

$$T^{\mu\nu} = \rho \begin{pmatrix} 1 & v^1 & v^2 & v^3 \\ v^1 & v^1v^1 & v^1v^2 & v^1v^3 \\ v^2 & v^2v^1 & v^2v^2 & v^2v^3 \\ v^3 & v^3v^1 & v^3v^2 & v^3v^3 \end{pmatrix} \sim \rho \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{115}$$

Then we have

$$\begin{aligned} T^{\mu\nu} - \frac{1}{2}Tg^{\mu\nu} &= \frac{1}{2}\rho \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\rho\delta^{\mu\nu}, \\ T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu} &= \frac{1}{2}\rho\delta_{\mu\nu}. \end{aligned} \tag{116}$$

Now let us consider the left hand side of Eq. (114),

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho{}_{\mu\nu} - \partial_\nu \Gamma^\rho{}_{\mu\rho} + \Gamma^\lambda{}_{\mu\nu}\Gamma^\rho{}_{\lambda\rho} - \Gamma^\lambda{}_{\mu\rho}\Gamma^\rho{}_{\lambda\nu}. \tag{117}$$

By using the weak field limit $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and then keeping only first order perturbations, one obtains

$$R_{\mu\nu} = \frac{1}{2}\eta^{\rho\sigma}\left(\partial_\rho\partial_\mu h_{\nu\sigma} + \partial_\rho\partial_\nu h_{\mu\sigma} - \partial_\rho\partial_\sigma h_{\mu\nu} - \partial_\nu\partial_\mu h_{\rho\sigma}\right). \tag{118}$$

Exercise: Show that by using the weak field limit $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and then keeping only first order perturbations, the Ricci tensor can be written as the above equation.

Imposing the static condition $\partial_0 h_{\mu\nu} = 0$, the component R_{00} can be written as

$$R_{00} = -\frac{1}{2}\eta^{ij}\partial_i\partial_j h_{00} = -\frac{1}{2}\nabla^2 h_{00} = \nabla^2\Phi, \tag{119}$$

where we have used Eq. (106). Substituting results into Eq. (114), one obtains

$$\nabla^2\Phi = \frac{1}{2}k\rho. \tag{120}$$

By comparing this equation to the Poisson equation $\nabla^2\Phi = 4\pi G\rho$, the constant k can be written as

$$k = 8\pi G. \tag{121}$$

Finally, the Einstein equation is completely constructed as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi G T_{\mu\nu}. \tag{122}$$

1.5 Lagrangian Formulation for General Relativity

There are various advantage points for Lagrangian formulation in field theories.

- Most of physical theories can be expressed in terms of Lagrangian formulation in which the equations of motion can be obtained by variational principle. Therefore, it is worthwhile to find Lagrangian formulation for General Relativity.

- It is convenient to identify the conserved quantities of the system from Lagrangian formulation. Moreover, one can relates the conserved quantities to the symmetries of the system.

- It is very useful to generalize the theory from Lagrangian formulation. For example, one can extend $U(1)$ gauge theory to non-Abelian gauge theory respected group symmetry $SU(2)$. This is an important issue in particle physics since this leads to a unification of electromagnetic interaction and weak interaction to form QED theory.

- It is very useful to generalize the theory to describe the extraordinary results of the experiments. This leads to an intensive study of modified gravity theories in order to describe the expanding acceleration of the universe.

- It provides some other points of view of the theory. For example, General Relativity can be interpreted as a theory of massless spin-2 in point of view of particle physics.

1.5.1 Review of Classical Field Theory

In order to study the dynamics of the system with a dynamical fields Φ^a in 4 dimensional spacetime where a denotes indices of each independent fields, it is useful to consider the action of this system as following form

$$S = \int d^4x \; \mathcal{L}(\Phi^a, \partial_\mu \Phi^a, \partial_\mu \partial_\nu \Phi^a, ...), \tag{123}$$

where $\mathcal{L}$ is Lagrangian density. Generally, $\mathcal{L}$ can be constructed under the symmetry we impose. From the fact that the physical quantities should be the same in all coordinates we use, this leads to the fact that $\mathcal{L}$ must be scalar quantities since scalar is invariant under coordinate transformation. Moreover, It is convenient to construct $\mathcal{L}$ in order to obtain the dimensionless action. This convenience comes from description of quantum field theory in path integral approach. One more requirement of construction of $\mathcal{L}$ is that it should be contained only up to first derivative terms, $\mathcal{L} = \mathcal{L}(\Phi^a, \partial_\mu \Phi^a)$. This requirement comes from the fact that most of equations of motion in physical field theory is the equation of motion up to their second derivative of the field. Note that this requirement may be violated since some of higher derivative terms in the Lagrangian density may leads to the equation of motion up to second derivative terms.

By using variational principle corresponding to the state that the physical system will evolve in such that the action is extremal, $\delta S = 0$, the equations of motion can be written as

$$\frac{\partial \mathcal{L}}{\partial \Phi^a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \right) + \partial_\mu \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \Phi^a)} \right) - ... = 0, \tag{124}$$

which is known as Euler-Lagrange equation.

- **Example.** Considering equation of motion of a massive real scalar field as follows

$$\partial_\mu \partial^\mu \phi - m^2 \phi = 0, \tag{125}$$

which known as Klein-Gordon equation, one can construct the action as

$$S = \int d^4x \left(-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right). \tag{126}$$

The first term corresponds to the kinetic term which is only one possible way to construct the scalar quantity from the first derivative of the field. The requirement of dimensionless action leads to the field have to be mass dimension. The factor 1/2 is just convention. To obtain the dimensionless one for the second term, this is only one possible term. The factor 1/2 is obtained by requiring of the suitable form the equation of motion. We can immediately check that this Lagrangian density satisfies the Klein-Gordon equation by substituting this Lagrangian density into Euler-Lagrange equation.

1.5.2 Classical Field Theory in Curved Spacetime

From GR I course, we have learned that scalar quantities in curved spacetime must constructed through the general metric, $g_{\mu\nu}$, instead of the Minkowski metric $\eta_{\mu\nu}$. For example, the scalar quantity $A_\mu A^\mu = g^{\mu\nu}A_\mu A_\nu$ is invariant under coordinate transformation instead of this quantity $\eta^{\mu\nu}A_\mu A_\nu$. Moreover, we also have learned that d^4x is not invariant under coordinate transformation while the appropriate volume element which is invariant under coordinate transformation must be $\sqrt{-g}d^4x$ where $g = \det g_{\mu\nu}$. Thus, the appropriate action should be in the form

$$S = \int d^4x\sqrt{-g}L, \qquad \text{where} \quad \mathcal{L} = \sqrt{-g}L. \tag{127}$$

Likewise, to have a derivative which is invariant under coordinate transformation, one must have a "covariant" derivative ∇_μ instead of partial derivative ∂_μ. Therefore, we obtain

$$L = L(\Phi^a, \nabla_\mu\Phi^a, \nabla_\mu\nabla_\nu\Phi^a, ...). \tag{128}$$

Consequently, the Euler-Lagrange equation in Eq. (124) will be modified as

$$\frac{\partial L}{\partial\Phi^a} - \nabla_\mu\left(\frac{\partial L}{\partial(\nabla_\mu\Phi^a)}\right) + \nabla_\mu\nabla_\nu\left(\frac{\partial L}{\partial(\nabla_\mu\nabla_\nu\Phi^a)}\right) - ... = 0, \tag{129}$$

- **Example 1. Simple real scalar field**

By using the same step with construction in flat spacetime, the simple (canonical) Lagrangian density can be written as

$$L = -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi). \tag{130}$$

Here we just change the partial derivative to the covariant derivative and we have to keep in mind that rising and lowering the indices is obtained by using the metric $g_{\mu\nu}$. The potential term is chosen as arbitrary function for generalization.

Substituting L from Eq.(130) into the Euler-Lagrange equation in Eq.

(129) we have

$$\begin{aligned}
\frac{\partial L}{\partial(\nabla_\mu\phi)} &= -\frac{1}{2}\frac{\partial}{\partial(\nabla_\mu\phi)}\left(g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi\right), \\
&= -\frac{1}{2}g^{\rho\sigma}\left(\left(\frac{\partial(\nabla_\rho\phi)}{\partial(\nabla_\mu\phi)}\right)\nabla_\sigma\phi + \nabla_\rho\phi\left(\frac{\partial(\nabla_\sigma\phi)}{\partial(\nabla_\mu\phi)}\right)\right), \\
&= -\frac{1}{2}\left(g^{\mu\sigma}\nabla_\sigma\phi + g^{\mu\rho}\nabla_\rho\phi\right), \\
&= -g^{\mu\sigma}\nabla_\sigma\phi. \\
\nabla_\mu\left(\frac{\partial L}{\partial(\nabla_\mu\phi)}\right) &= \nabla_\mu\left(g^{\mu\sigma}\nabla_\sigma\phi\right), \\
&= -g^{\mu\sigma}\nabla_\mu\nabla_\sigma\phi \quad = - \quad \nabla_\mu\nabla^\mu\phi. \qquad (131)
\end{aligned}$$

Finally, the equation of motion can be written as

$$\nabla_\mu\nabla^\mu\phi - V_{,\phi} = 0, \qquad (132)$$

where $V_{,\phi}$ denotes $\frac{dV}{d\phi}$. Note that $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ is covariant scalar since $\nabla_\mu\phi = \partial_\mu\phi$. However, for convenient, people write it in the form of covariant derivative. It just remind ourself during calculation that we are performing in curved spacetime. From the equation of motion above, one can recover Klein-Gordon equation in curved spacetime by setting the potential as $V = \frac{1}{2}m^2\phi^2$.

- **Example 2. Gauge field**

Now we consider vector field which obey gauge symmetry while the gauge transformation can be written as

$$A_\mu \to A'_\mu = A_\mu + \nabla_\mu\Psi, \qquad (133)$$

where Ψ is a scalar field. For kinetic terms, they must be gauge invariant. we know that the field strength tensor $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ are gauge invariant and also contain first derivative the gauge field. Therefore, it is useful to construct scalar quantities from the field strength tensor. Consequently, the kinetic term can be written as

$$L_K = \alpha F_{\mu\nu}F^{\mu\nu}, \qquad (134)$$

where α is just a proportional constant determined later. For the mass term or potential term, it is found that the scalar quantity $m^2 A_\mu A^\mu$ violates gauge symmetry. Note that this is the reason why photon is massless. In order to obtain Maxwell equations, we can add the source into the theory. Finally the general action of the gauge field can be written as

$$S = \int d^4x\sqrt{-g}\left(\alpha F_{\mu\nu}F^{\mu\nu} - A_\mu j^\mu\right), \tag{135}$$

where j^μ is four current density. Substituting Lagrangian density from Eq. (135) into Euler-Lagrange equation, we have

$$\begin{aligned}
\frac{\partial L}{\partial(\nabla_\mu A_\nu)} &= \alpha g^{\alpha\rho}g^{\beta\sigma}\frac{\partial}{\partial(\nabla_\mu A_\nu)}\left(F_{\alpha\beta}F_{\rho\sigma}\right), \\
&= \alpha g^{\alpha\rho}g^{\beta\sigma}\left(\frac{\partial F_{\alpha\beta}}{\partial(\nabla_\mu A_\nu)}F_{\rho\sigma} + \frac{\partial F_{\rho\sigma}}{\partial(\nabla_\mu A_\nu)}F_{\alpha\beta}\right), \\
&= 2\alpha g^{\alpha\rho}g^{\beta\sigma}\left(F_{\rho\sigma}\frac{\partial F_{\alpha\beta}}{\partial(\nabla_\mu A_\nu)}\right), \\
&= 2\alpha g^{\alpha\rho}g^{\beta\sigma}\left(F_{\rho\sigma}\frac{\partial(\nabla_\alpha A_\beta - \nabla_\beta A_\alpha)}{\partial(\nabla_\mu A_\nu)}\right), \\
&= 2\alpha g^{\alpha\rho}g^{\beta\sigma}\left(F_{\rho\sigma}(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha)\right), \\
&= 4\alpha F^{\mu\nu}. \quad \Rightarrow \nabla_\mu\left(\frac{\partial L}{\partial(\nabla_\mu A_\nu)}\right) = 4\alpha\nabla_\mu F^{\mu\nu}.
\end{aligned} \tag{136}$$

Also, we have

$$\frac{\partial L}{\partial A_\nu} = -\frac{\partial A_\rho j^\rho}{\partial A_\nu} = -\delta^\nu_\rho j^\rho = -j^\nu. \tag{137}$$

And then, the equation of motion can be written as

$$4\alpha\nabla_\mu F^{\mu\nu} + j^\nu = 0, \tag{138}$$

while the Maxwell equations are

$$\nabla_\mu F^{\mu\nu} = \mu_0 j^\nu. \tag{139}$$

Therefore, we obtain the parameter α as $\alpha = -1/(4\mu_0)$. Finally, The action for electromagnetic theory can be rewritten as

$$S = \int d^x\sqrt{-g}\left(-\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - A_\mu j^\mu\right), \tag{140}$$

1.5.3 Einstein-Hilbert Action

Now we are in position to find the action for general relativity. In general relativity, the dynamical field of the theory is the metric $g_{\mu\nu}$. In order to find the Lagrangian density of the theory, let us consider Lagrangian density as $L_g = f(g_{\mu\nu}, \partial_\rho g_{\mu\nu})$. For the kinetic term, it may be constructed from $\partial_\rho g_{\mu\nu}$. However, $\partial_\rho g_{\mu\nu}$ is not covariant quantity while the covariant form of the first is $\nabla_\rho g_{\mu\nu}$. Unfortunately, One cannot use may use $\nabla_\rho g_{\mu\nu}$ to construct the Lagrangian density since it always vanish, $\nabla_\rho g_{\mu\nu} = 0$. Therefore, one may proceed our attention to the second derivative, $\partial_\rho\partial_\sigma g_{\mu\nu}$ but this quantity is not covariant. The covariant quantity which contains the first and second derivative of the metric is Reimannian curvature tensor, $R_{\mu\nu\rho\sigma}$. However, as we mentioned before, Lagrangian density must be covariant scalar. The only one covariant scalar constructed from $R_{\mu\nu\rho\sigma}$ is the Ricci scalar R. Note that the term like $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ and $R_{\mu\nu}R^{\mu\nu}$ are also covariant scalar but they contain the higher than second order derivative terms. Let us try first to use this scalar to be the Lagrangian density for the gravity. Therefore, the action can be expressed as

$$S_g = \int d^4x\sqrt{-g}\, R, \tag{141}$$

where this action is known as Einstein-Hilbert action. In order to see that this action correspond to the Einstein field equation or not, let us use variational principle to find the equation of motion of this action. Since R is very complicated differential form of the the metric, we will use direct variational principle to the action instead of use the Euler-Lagrange equation. Note that the calculation by using Euler-Lagrange equation is very lengthy but straightforwardly. Varying the action with respect to the metric we have

$$\delta S_g = \int d^4x\delta(\sqrt{-g}\, g^{\mu\nu}R_{\mu\nu}) = \int d^4x\left[\underbrace{\delta\sqrt{-g}\, g^{\mu\nu}R_{\mu\nu}}_{\delta S_{g_3}} + \underbrace{\sqrt{-g}\, \delta g^{\mu\nu}R_{\mu\nu}}_{\delta S_{g_1}} + \underbrace{\sqrt{-g}\, g^{\mu\nu}\delta R_{\mu\nu}}_{\delta S_{g_2}}\right] \tag{142}$$

From the identity

$$\begin{aligned} \ln(\det M) &= \mathrm{Tr}(\ln M), \qquad \det M \neq 0, \\ \ln(g) &= \mathrm{Tr}(\ln[g_{\mu\nu}]). \end{aligned} \tag{143}$$

Then, consider the variation

$$\begin{aligned} \delta\ln(g) &= \delta\mathrm{Tr}(\ln[g_{\mu\nu}]), \\ \frac{1}{g}\delta g &= (g^{-1})_{\mu\nu}\delta g_{\mu\nu}, \\ \delta g &= g\, g^{\mu\nu}\delta g_{\mu\nu}, \\ \delta\sqrt{-g} &= \frac{1}{2\sqrt{-g}}\delta(-g) = -\frac{1}{2\sqrt{-g}}g\, g^{\mu\nu}\delta g_{\mu\nu}, \\ &= \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu}. \end{aligned} \tag{144}$$

From

$$\begin{aligned} g_{\mu\rho}g^{\rho\nu} &= \delta^\nu_\mu, & (145)\\ \delta g_{\mu\rho}g^{\rho\nu} + g_{\mu\rho}\delta g^{\rho\nu} &= 0, \\ g^{\rho\nu}\delta g_{\mu\rho} &= -g_{\mu\rho}\delta g^{\rho\nu}, \\ \delta g_{\mu\nu} &= -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}. & (146)\end{aligned}$$

Then, Eq. (144) can be expressed as

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\; g_{\mu\nu}\delta g^{\mu\nu}. \tag{147}$$

Substituting this result into (142), one obtains

$$\begin{aligned} \delta S_{g_1} + \delta S_{g_3} &= \int d^4x\sqrt{-g}\;\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu}, & (148)\\ \delta S_{g_2} &= \int d^4x\sqrt{-g}\; G_{\mu\nu}\delta g^{\mu\nu}. \end{aligned}$$

From these two terms, one properly obtain Einstein tensor which is the left hand side of Einstein field equation. However, we have to consider the second term which is proportional to $\delta R_{\mu\nu}$. In order to obtain $\delta R_{\mu\nu}$, one needs to know the form of $\delta\Gamma^\rho_{\mu\nu}$. Let us start with varying the connection as following

$$\begin{aligned} \delta\Gamma^\rho_{\mu\nu} &= \frac{1}{2}\delta g^{\rho\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) + g^{\rho\sigma}\left(\partial_\mu\delta g_{\sigma\nu} + \partial_\nu\delta g_{\sigma\mu} - \partial_\sigma\delta g_{\mu\nu}\right) & (149)\\ &= \frac{1}{2}g^{\rho\sigma}\left(\nabla_\mu\delta g_{\sigma\nu} + \nabla_\nu\delta g_{\sigma\mu} - \nabla_\sigma\delta g_{\mu\nu}\right). & (150)\end{aligned}$$

The detail calculation from the fist to the second line is left for the student in the Exercise. From the above expression, one can see that even though the connection $\Gamma^\rho_{\mu\nu}$ is not tensor, the variation $\delta\Gamma^\rho_{\mu\nu}$ is. Then, we now move to calculate of the variation of the Ricci tensor $\delta R_{\mu\nu}$ as following

$$\begin{aligned} \delta R_{\mu\nu} &= \delta\left(\partial_\mu\Gamma^\rho_{\nu\rho} - \partial_\rho\Gamma^\rho_{\nu\mu} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu} - \Gamma^\rho_{\rho\sigma}\Gamma^\sigma_{\mu\nu}\right)\\ &= \partial_\mu\delta\Gamma^\rho_{\nu\rho} - \partial_\rho\delta\Gamma^\rho_{\nu\mu} + \delta\Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu} + \Gamma^\rho_{\mu\sigma}\delta\Gamma^\sigma_{\rho\nu} - \delta\Gamma^\rho_{\rho\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\rho_{\rho\sigma}\delta\Gamma^\sigma_{\mu\nu},\\ &= \partial_\mu\delta\Gamma^\rho_{\nu\rho} + \Gamma^\rho_{\mu\sigma}\delta\Gamma^\sigma_{\rho\nu} - \Gamma^\sigma_{\mu\nu}\delta\Gamma^\rho_{\rho\sigma} - (\partial_\rho\delta\Gamma^\rho_{\nu\mu} - \delta\Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu} + \Gamma^\rho_{\rho\sigma}\delta\Gamma^\sigma_{\mu\nu}),\\ &= \partial_\mu\delta\Gamma^\rho_{\nu\rho} + \Gamma^\rho_{\mu\sigma}\delta\Gamma^\sigma_{\rho\nu} - \Gamma^\sigma_{\mu\nu}\delta\Gamma^\rho_{\rho\sigma} - \Gamma^\sigma_{\mu\rho}\delta\Gamma^\rho_{\nu\sigma} - (\partial_\rho\delta\Gamma^\rho_{\nu\mu} - \delta\Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu} + \Gamma^\rho_{\rho\sigma}\delta\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\mu\rho}\delta\Gamma^\rho_{\nu\sigma}),\\ &= \nabla_\nu\delta\Gamma^\rho{}_{\mu\rho} - \nabla_\rho\delta\Gamma^\rho{}_{\mu\nu}. & (151)\end{aligned}$$

Therefore, the second term in Eq. (142) can be expressed as

$$\begin{aligned}
\delta S_{g_2} &= \int d^4x\sqrt{-g}\; g^{\mu\nu}\left(\nabla_\nu \delta\Gamma^\rho{}_{\mu\rho} - \nabla_\rho \delta\Gamma^\rho{}_{\mu\nu}\right), \\
&= \int d^4x\sqrt{-g}\; \nabla_\nu \left(\delta^\nu_\sigma \nabla_\rho \delta g^{\rho\sigma} - g_{\rho\sigma} g^{\nu\alpha} \nabla_\alpha \delta g^{\rho\sigma}\right), \\
&= \int d^4x\sqrt{-g}\; \nabla_\nu V^\nu. \qquad (152)
\end{aligned}$$

In order to take this contribution to be zero, one requires that the vector V^ν must vanish at the surface. This is not normal situation we found in the literature since we require the first derivative of the variation of the dynamical field to be zero at the surface,$\nabla_\mu \delta g^{\rho\sigma}$, instead of its variation $\delta g^{\rho\sigma}$. This is the price we pay for adding second derivative terms into the Lagrangian density. Another way to consistently take this term to be zero is including additional term to cancel out this term. The additional term are usually proportional to the surface term and the Lagrangian density is proportional to the extensive curvature of the surface [**?**]. The interpretation physical properties of this term still in active area of gravitational and cosmological researches. Student who especially interested of this subject can study further in the reference and also can provide a results as a report and presentation of this course. Finally, we have the variation as

$$\delta S_g = \int d^4x\sqrt{-g}\; G_{\mu\nu}\delta g^{\mu\nu} = 0, \Rightarrow G_{\mu\nu} = 0. \qquad (153)$$

This is the Einstein equation in the vacuum. In order to obtain the Einstein equation with source one has to include the action for matter field into our consideration.

1.5.4 Energy-Momentum Tensor

If the matter is also considered, there exists the action for the matter part,

$$S = S_g + S_m = \int d^4x \left(\frac{1}{2\kappa}\mathcal{L}_{EH} + \mathcal{L}_m\right), \qquad (154)$$

where S_m denotes the action for matter contents, κ is a constant and the factor number 2 is just convention. From Eq. (153), the variation of the action can be written as

$$\begin{aligned}
\delta S &= \int d^4x\left(\frac{1}{2\kappa}\delta\mathcal{L}_{EH}+\delta\mathcal{L}_m\right),\\
&= \int d^4x\left(\frac{1}{2\kappa}\sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu}+\delta\mathcal{L}_m\right),\\
&= \int d^4x\sqrt{-g}\delta g^{\mu\nu}\frac{1}{2\kappa}\left(G_{\mu\nu}+\frac{2\kappa}{\sqrt{-g}}\frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}}\right)=0.
\end{aligned}$$

In order to obtain Einstein equation, one can define the energy momentum tensor of the matter as

$$\begin{aligned}
T_{\mu\nu} &\equiv \frac{-2}{\sqrt{-g}}\frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}}=\frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}},\\
&= \frac{-2}{\sqrt{-g}}\left(\sqrt{-g}\frac{\delta L_m}{\delta g^{\mu\nu}}+L_m\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\right),\\
&= 2\left(-\frac{\delta L_m}{\delta g^{\mu\nu}}+\frac{1}{2}L_m g_{\mu\nu}\right), \qquad (155)
\end{aligned}$$

where

$$\kappa\equiv\frac{8\pi G}{c^4}. \qquad (156)$$

In classical field theory, there is a similar definition of the energy momentum tensor called ”canonical energy momentum tensor” corresponding to Noether current. This Noether current is associated the translational symmetry of the Lagrangian and can be defined as

$$S^{\mu\nu} \equiv \frac{\delta\mathcal{L}}{\delta(\partial_\mu\Phi^i)}\partial^\nu\Phi^i-\eta^{\mu\nu}\mathcal{L}. \qquad (157)$$

where Φ^i are field we are considering. This definition is less useful than we defined before since it cannot be generalized to one in curve spacetime. Moreover, our definition is manifestly symmetric.

In the case that the matter is only a scalar field,

$$L_m=L_\phi=-\frac{1}{2}g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi-V(\phi), \qquad (158)$$

then the energy momentum tensor for the scalar field can be found as follows,

$$\begin{aligned}
\frac{\delta L_\phi}{\delta g^{\mu\nu}} &= -\frac{1}{2}\frac{\delta g^{\rho\sigma}}{\delta g^{\mu\nu}}\nabla_\rho\phi\nabla_\sigma\phi,\\
&= -\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi,\\
T_{\mu\nu} &= \nabla_\mu\phi\nabla_\nu\phi+\left(\frac{1}{2}g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi+V(\phi)\right)g_{\mu\nu}. \qquad (159)
\end{aligned}$$

1.5.5 Infinitesimal general coordinate transformation

The objective of this section is to see the relation between the conservation of matter and the variation under IGCT. This transformation for the coordinate, x^μ reads

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu, \tag{160}$$

where ξ^μ is a infinitesimal parameter (its value is very small). The variation of action for matter is written as

$$\delta S_m = \int \mathrm{d}^4x\sqrt{-g}T_{\mu\nu}\delta g_{\mu\nu}. \tag{161}$$

Considering $\delta g_{\mu\nu}$ under IGCT,

$$g_{\mu\nu} \to g'^{\mu\nu} = g^{\mu\nu} + \delta g^{\rho\sigma}. \tag{162}$$

Under the full GCT, the metric transforms as follows

$$g'^{\mu\nu}(x'^\alpha) = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma}\, g^{\rho\sigma}(x^\alpha). \tag{163}$$

Putting (160) and keeping to the first order of ξ^μ,

$$\begin{aligned} g'^{\mu\nu}(x'^\alpha) &= (\delta^\mu_\rho + \partial_\rho\xi^\mu)(\delta^\nu_\sigma + \partial_\sigma\xi^\nu)g^{\rho\sigma}(x^\alpha), \\ &\simeq g^{\mu\nu}(x^\alpha) + g^{\rho\nu}(x^\alpha)\partial_\rho\xi^\mu + g^{\rho\mu}(x^\alpha)\partial_\rho\xi^\nu, \\ &= g^{\mu\nu}(x^\alpha) + 2g^{\rho(\mu}\partial_\rho\xi^{\nu)}. \end{aligned} \tag{164}$$

See that $g'^{\mu\nu}(x'^\alpha)$ and $g^{\mu\nu}(x^\alpha)$ are considered in different points. In order to obtain the quantities on the same point, doing the Taylor's expansion of $g'^{\mu\nu}(x'^\alpha)$ about the point x^α as

$$\begin{aligned} g'^{\mu\nu}(x'^\alpha) &= g'^{\mu\nu}\Big|_{x'=x} + \partial_\rho g'^{\mu\nu}\Big|_{x'=x}(x'^\rho - x^\rho) + \dots, \\ &\simeq g^{\mu\nu}(x^\alpha) + \partial_\rho g^{\mu\nu}(x^\alpha)\,\xi^\rho. \end{aligned} \tag{165}$$

Hence, the variation of metric is

$$\begin{aligned} \delta g^{\mu\nu}(x^\alpha) &= g'^{\mu\nu}(x^\alpha) - g^{\mu\nu}(x^\alpha), \\ &\simeq -\xi^\rho\partial_\rho g^{\mu\nu} + 2g^{\rho(\mu}\partial_\rho\xi^{\nu)}. \end{aligned} \tag{166}$$

Let us compute,

$$\begin{aligned} \nabla^\mu\xi^\nu &= g^{\mu\rho}\nabla_\rho\xi^\nu, \\ &= g^{\mu\rho}(\partial_\rho\xi^\nu + \Gamma^\nu_{\rho\sigma}\xi^\sigma), \end{aligned} \tag{167}$$

$$\begin{aligned} \to \quad \nabla^\mu\xi^\nu + \nabla^\nu\xi^\mu &= 2g^{\rho(\mu}\partial_\rho\xi^{\nu)} + 2g^{\rho(\mu}\Gamma^{\nu)}_{\rho\sigma}\xi^\sigma, \\ \to \quad 2g^{\rho(\mu}\partial_\rho\xi^{\nu)} &= \nabla^\mu\xi^\nu + \nabla^\nu\xi^\mu - 2g^{\rho(\mu}\Gamma^{\nu)}_{\rho\sigma}\xi^\sigma, \end{aligned} \tag{168}$$

and

$$\begin{aligned}
\nabla_\rho g^{\mu\nu}\xi^\rho &= 0 = \partial_\rho g^{\mu\nu}\xi^\rho + \Gamma^\mu_{\rho\sigma} g^{\sigma\nu}\xi^\rho + \Gamma^\nu_{\rho\sigma} g^{\mu\sigma}\xi^\rho, && (169)\\
\rightarrow \quad -\xi^\rho\partial_\rho g^{\mu\nu} &= 2g^{\sigma(\mu}\Gamma^{\nu)}_{\rho\sigma}\xi^\rho. && (170)
\end{aligned}$$

Plugging (168) and (170) into (166),

$$\begin{aligned}
\delta g^{\mu\nu} &= \cancel{2g^{\sigma(\mu}\Gamma^{\nu)}_{\rho\sigma}\xi^\rho} + \nabla^\mu\xi^\nu + \nabla^\nu\xi^\mu - \cancel{2g^{\rho(\mu}\Gamma^{\nu)}_{\rho\sigma}\xi^\sigma},\\
&= \nabla^\mu\xi^\nu + \nabla^\nu\xi^\mu,\\
&= 2\nabla^{(\mu}\xi^{\nu)}. && (171)
\end{aligned}$$

The variation of action for matter in (161) becomes

$$\delta S_m = \int \mathrm{d}^4x\sqrt{-g}T_{\mu\nu}2\nabla^{(\mu}\xi^{\nu)}. \tag{172}$$

Since $T_{\mu\nu}$ is symmetric, we can write

$$\begin{aligned}
\delta S_m &= \int \mathrm{d}^4x\sqrt{-g}T_{\mu\nu}\nabla^\mu\xi^\nu,\\
&= \int \mathrm{d}^4x\sqrt{-g}\left[\nabla^\mu(T_{\mu\nu}\xi^\nu) - \nabla^\mu T_{\mu\nu}\right],\\
&= \int \mathrm{d}^4x\sqrt{-g}\left(-\xi^\nu\nabla^\mu T_{\mu\nu}\right) + \left(T_{\mu\nu}\xi^\nu\right)\Big|_{\text{boundary}}. && (173)
\end{aligned}$$

Let the infinitesimal parameter vanishes at boundary and we consider $\xi^\mu \neq 0$ in volume. So, we obtain

$$\delta S_m = 0, \quad \rightarrow \quad \nabla^\mu T_{\mu\nu} = 0. \tag{174}$$

We can conclude that the conservation for matter is obtained from the variation of action for matter under IGCT.

2 Cosmological Principle and FLRW metric

We have learned so far the geometry of the simple astronomical objects by using General Relativity. It is found that some distinguished observational predictions of General Relativity are compatible with observational data. This makes General Relativity become the one of fundamental theory of nature. It is worthwhile to test General Relativity with cosmological objects. This can be evaluated by considering geometry of the universe. The basic concept for applying General Relativity to describe the evolution of the universe is main content of this chapter.

2.1 Cosmological principle

Due to nonlinearity of Einstein field equation, it is not easy to obtain general solutions of this equation. So far, we have learned that it is possible to extract some solutions from this equation by imposing reasonable symmetry to the theory. For example, the Schwarzschild solution can be obtained by solving Einstein field equation with static and spherical symmetry. In cosmology, the symmetries of spacetime are provided by cosmological principle. Cosmological principle is a principle based on the assumption as following ”the universe is isotropic and homogeneous in three-space”. This principle provides a smooth or uniform universe at large scale. It is important to note that this principle is compatible with observational data, specifically Cosmic Microwave Background (CMB) radiation data. Therefore this assumption (cosmological principle) provide a reasonable symmetries to study geometry and dynamics of the universe by using General Relativity.

Since the cosmological principle provides us the symmetry only in three-dimensional space. Therefore, one can sprite the spacetime into product space as follows

$$\underbrace{\mathbb{R}}_{\text{real}} \times \underbrace{\Sigma}_{\text{3-space}}, \tag{175}$$

where $\mathbb{R}$ represents (real) time and Σ represents a homogeneous and isotropic 3-space. Therefore, the spacetime can be treated as a series of non-interacting spacelike hypersurface labeling by universal time parameter as shown in figure 16. The the 3-dimensional hypersurface which is homogeneous and isotropic will be obeyed translational and rotational invariants respectively. This corresponds to maximally symmetric space. This kind of space will provide us the simple way to find the general form of the metric. We will discuss the maximally symmetric space in next subsection in detail.

2.2 Maximally symmetric space

By definition, the maximally symmetric space is a space which has number of degrees of freedom of the metric is equal to number of isometry of the metric. Note that isometry is a symmetry of the metric which is not necessary to be a physical symmetry. To understand clearly, let us consider n-dimensional flat Euclidean space, $\mathbb{R}^n$. This space will be described by a range-2 $(n \times n)$ symmetric metric $g_{\mu\nu}$. The number of degrees of freedom of this metric is

$$n(n+1)/2. \tag{176}$$

Considering the space with has rotational and translational invariants, we have

$$\left.\begin{array}{ll} \text{translation} & \Rightarrow n \text{ degrees of freedom} \\ \text{rotation} & \Rightarrow \frac{n}{2}\left(n-1\right) \text{ degrees of freedom} \end{array}\right\} n+\frac{n}{2}\left(n-1\right)=\frac{n}{2}\left(n+1\right) \quad (177)$$

We can see that the number of degree s of freedom of the metric is equal to the number of the isometry of the metric. This corresponds to the maximally symmetric space. Note that, in n dimensions, we can rotate x axis to one of the other $n-1$ axes and the next axis can be rotated to other $n-2$ axes, which means all of the axes can be rotated

$$(n-1)+(n-2)+(n-3)+\ldots+1=\frac{n}{2}\left(n-1\right) \quad \text{ways.} \quad (178)$$

It makes our sense that the more number of isometry, the fewer functions needed to specify the properties of the spacetime. One of important consequence of the maximally symmetric space is that there is only one number needed to specify. This number is independent of the coordinates and corresponds to the curvature K. Maximally symmetric space will provide the fact that $R_{\rho\sigma\mu\nu}$ is the same everywhere. Since the curvature tensor depends on the metric, by using the symmetry of the indices, one has

$$R_{\rho\sigma\mu\nu}\propto g_{\rho\mu}g_{\sigma\nu}-g_{\rho\nu}g_{\sigma\mu}=K\left(g_{\rho\mu}g_{\sigma\nu}-g_{\rho\nu}g_{\sigma\mu}\right). \quad (179)$$

Consequently, the Ricci tensor and Ricci scalar can be written as

$$\begin{aligned} R_{\sigma\nu} &= g^{\rho\mu}R_{\rho\sigma\mu\nu}=g^{\rho\mu}K\left(g_{\rho\mu}g_{\sigma\nu}-g_{\rho\nu}g_{\sigma\mu}\right), \\ &= K\left(\delta^{\mu}_{\mu}g_{\sigma\nu}-\delta^{\mu}_{\nu}g_{\sigma\mu}\right)=K\left(n-1\right)g_{\sigma\nu}, \quad (180) \\ R &= g^{\sigma\nu}R_{\sigma\nu}=g^{\sigma\nu}K\left(n-1\right)g_{\sigma\nu}=K\left(n-1\right)\delta^{\nu}_{\nu}=Kn\left(n-1\right). \quad (181) \end{aligned}$$

Finally, one find that the number corresponding to the space curvature can be written in terms of Ricci scalar as follows

$$K = \frac{R}{n\left(n-1\right)}. \quad (182)$$

This is a main result follows from the maximally symmetric space. We will use this result in next subsection in order to find the general form of the metric describing the dynamics of the universe.

2.3 Friedmann-Lemaitre-Robertson-Walker metric

Considering the homogeneous and isotropic hypersurface, the general form of the line element can be written as

$$d\sigma^2=g_{ij}dx^idx^j. \quad (183)$$

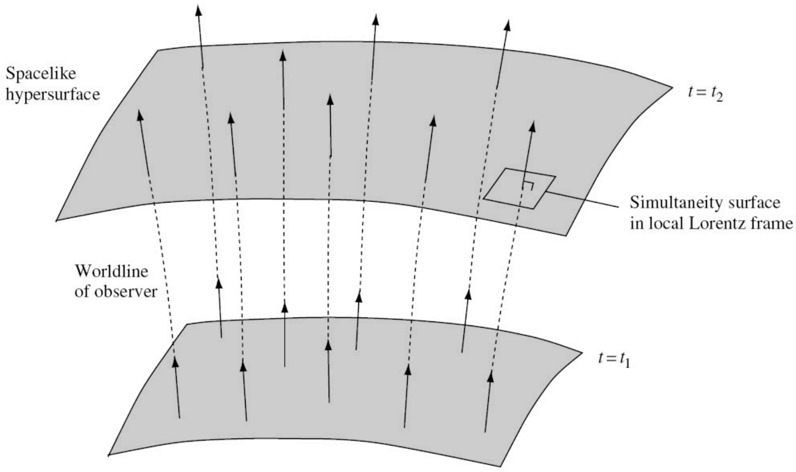


Figure 16: Non-interacting hypersurface parametized by universal time parameter.

In order to illustrate the form of the metric, one may consider three points triangle in this space. For isotropy, the triangle must be the same for all time. From this argument one find that each hypersurface will be different only by a overall factor $d\sigma_2^2 = S^2(x^k,t)d\sigma_1^2 = S^2(x^k,t)g_{ij}dx^i dx^j$. For homogeneity, it will be satisfied only if the overall factor must be independent of the coordinates x^k. Then most general form of the homogeneous and isotropic hypersurface can be written as

$$d\sigma^2 = S^2(t)\gamma_{ij}dx^i dx^j, \tag{184}$$

where h_{ij} is a component of the metric in the original hypersurface depending only on spatial coordinates x^k.The coordinate x^k are the comoving coordinates while the observers in these coordinate will not see any dynamics of the objects called comoving observers. Therefore, the most general form of the metric in this spacetime can be written as

$$ds^2 = -dt^2 + S^2(t)\gamma_{ij}dx^i dx^j, \tag{185}$$

where the term proportional to $dtdx^i$ disappears due to the requirement of non-interacting hypersurface. Now we are in the position to find the form of the metric h_{ij}. Let us consider the subspace defined as following

$$d\sigma^2 = \gamma_{ij}dx^i dx^j = e^{2B}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{186}$$

where we have used the coordinate transformation to eliminate the other function. Note that for isotropy one find that there are only two independent functions to specify the metric. However, from the general coordinate invariant, we can use coordinate transformation to eliminate one of them. By using the properties of the maximally symmetric space as discussed in the previous subsection one has

$$\begin{aligned}
R_{\mu\nu} &= K\,(n-1)\,g_{\mu\nu} = -2Kg_{\mu\nu}, \\
R_{rr} &= \frac{2}{r}B' = 2Ke^{2B}, && (187)\\
R_{\theta\theta} &= -e^{-2B}\,(1-rB') - 1 = 2Kr^2. && (188)
\end{aligned}$$

We leave detail calculation for Γ^i_{jk} and R_{ij} in the Exercise. From the Eq. (187) one has

$$\begin{aligned}
B' &= Kre^{2B}, \\
\int e^{-2B}dB &= K\int rdr, \\
e^{-2B} &= -Kr^2 + C, && (189)
\end{aligned}$$

where C is an integrate constant. Substitute the Eq. (189) into the Eq. (188) we have

$$\begin{aligned}
e^{-2B}\,(1-rB') - 1 &= -2Kr^2, \\
e^{-2B}\left(1 - r^2Ke^{2B}\right) - 1 &= -2Kr^2, \\
e^{-2B} - r^2K - 1 &= -2Kr^2, \\
-\cancel{Kr^2} + C - \cancel{r^2K} - 1 &= -\cancel{2Kr^2}, \\
C &= 1. && (190)
\end{aligned}$$

Thus, one has

$$e^{-2B} = 1 - Kr^2, \tag{191}$$

and the line element in Eq. (185) will then be

$$ds^2 = -dt^2 + S^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]. \tag{192}$$

To classify the geometry of this spacetime, one can consider the value of the curvature K. One find that the geometry of the spacetime can be distinguished by the sign of the curvature $K > 0$, $K < 0$ and $K = 0$. To see explicitly, one may redefine the curvature in terms of the dimensionless one as follows

$$k = \frac{K}{|K|}, \qquad \bar{r} = |K|^{\frac{1}{2}}r, \qquad R^2(t) = \frac{S^2(t)}{|K|}. \tag{193}$$

Therefore, the line element in Eq. (192) can be expressed as

$$ds^2 = -dt^2 + R^2(t)\left[\frac{d\bar{r}^2}{1-k\bar{r}^2} + \bar{r}^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \tag{194}$$

where the curvature of the maximally symmetric space can be classified very simply as

$$\begin{aligned} k &= -1 \quad \rightarrow \quad \text{open}, \\ k &= 0 \quad \rightarrow \quad \text{flat}, \\ k &= 1 \quad \rightarrow \quad \text{closed}. \end{aligned} \tag{195}$$

It is important to note that the metric explicitly obeys the rotational invariant. This is is also by construction of the metric we have used. For the translational invariant, it is a hidden symmetry. One cannot see explicitly from the metric. However, we can see that one can choose the origin of the radial coordinate completely arbitrary. Therefore it means that the metric contains translational invariant.

In order to obtain the geometric properties of FLRW metric, let us consider the line element of the three-dimensional hypersurface as follows

$$d\sigma^2 = \frac{d\bar{r}^2}{1-k\bar{r}^2} + \bar{r}^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{196}$$

By using the coordinate transform such that

$$d\chi = \frac{d\bar{r}}{\sqrt{1-k\bar{r}^2}}, \tag{197}$$

the line element in Eq. (196) can be written as

$$d\sigma^2 = d\chi^2 + \underline{S}^2(\chi)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{198}$$

where $\underline{S}$ can be classified according to different kinds of curvature as

$$\underline{S}^2(\chi) = \sinh^2\chi \qquad \text{for } k = -1, \tag{199}$$
$$\underline{S}^2(\chi) = \chi^2 \qquad \text{for } k = 0, \tag{200}$$
$$\underline{S}^2(\chi) = \sin^2\chi \qquad \text{for } k = 1. \tag{201}$$

By embedding these three-dimensional surface into four-dimensional Euclidean space, one find that for $k = 1$ the surface is the three-dimensional sphere, for $k = 0$ the surface is flat and for $k = -1$ the surface is hyperboloid. This is the reason why we call closed, flat and open geometry for

$k = 1$, $k = 0$ and $k = -1$ respectively. We leave the explicit calculations for this embedding procedure in Exercise.

From Eq. (194), the radial coordinate $\bar{r}$ is dimensionless and the scale factor $\underline{S}$ is length dimension. It is convenient to consider the scale factor with dimensionless since it determines the how much the three-surface will be magnified. Conventionally, most of cosmologists redefine the radial coordinate $\bar{r}$, the scale factor and the curvature as follows

$$a(t) = \frac{R(t)}{R_0}, \quad r = R_0\bar{r}, \quad \kappa = \frac{k}{R_0^2}. \tag{202}$$

Therefore the line element in Eq. (194) can be written in terms of new variables as

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \tag{203}$$

The metric corresponding to this line element is commonly use in cosmology and the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric is usually referred to this metric. It is important to note that the curvature κ is now dimensionfull and can be chosen as arbitrary number. The geometry of the spacetime can be classified by $\kappa > 0$, $\kappa = 0$ and $\kappa < 0$. However, It is convenient to choose this number as $\kappa = 1$, $\kappa = 0$ and $\kappa = -1$ for closed, flat and open geometry of the three-dimensional hypersurface.

Exercise:

1. From the line element $d\sigma^2 = \gamma_{ij}dx^i dx^j = e^{2B}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$, find non-zero components of Γ^i_{jk} and R_{ij}.

2. By embedding three-dimensional hypersurface $d\sigma^2 = \frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$ into four-dimensional Euclidean space, show that for $k = 1$, $k = 0$ and $k = -1$ the geometry of the space corresponds to sphere, flat and hyperboloid respectively.

2.4 Energy momentum tensor

In order to study dynamics of the universe, one has to use the Einstein field equation with FLRW metric. In the left hand side of the Einstein equation, there is only one function to determine which is the scale factor $a(t)$ and only one number which is the curvature κ. The scale factor will provide us the evolution behavior of the universe and the curvature will provide us the geometry of the three-dimensional hypersurface. For the right hand side of the Einstein equation,one needs to find the form of the fluid to satisfy the homogeneity and isotropy of the universe.

2.4.1 Perfect fluid

The perfect fluid is the fluid with out viscosity and heat transfer between its elements in its rest frame. Since there are no heat transfers, the components T^{0i} will vanish. This satisfies the same requirement for the metric where $g_{0i} = 0$. Moreover, no viscosity leads to no shear between elements of the fluid, so that the components $T^{ij}, (i \neq j)$ vanish. This also satisfies the same requirement for the metric where $g_{ij} = 0$ for $i \neq j$. As a result, for the perfect fluid there are two functions to determine which are energy density, ρ, and pressure, p, of the fluid. Note that the pressure p serves as the the pressure in every direction satisfying isotropic condition. Moreover, to satisfy the homogeneity and isotropy, one has to require that the energy density and pressure must be independent of the spatial coordinates, so that one obtains $\rho = \rho(t)$ and $p = p(t)$. As a result, the EMT can be written in the matrix form as

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{204}$$

Note that this form of the EMT is valid only in its rest frame. One needs a covariant form of the EMT in order to use it in the field equation. Let us find this covariant form by using the Lorentz transformation with Lorentz matrix as

$$\Lambda^i_j = \delta^i_j + \frac{\gamma - 1}{v^2} v^i v_j, \quad \Lambda^i_0 = \gamma v^i, \quad \Lambda^0_0 = \gamma. \tag{205}$$

By applying these to the energy momentum above, one obtains

$$T^{ij} = p\delta^{ij} + v^i v^j \gamma^2 (\rho + p), \tag{206}$$

$$T^{0i} = v^i \gamma^2 (\rho + p), \tag{207}$$

$$T^{00} = (\gamma - 1)p + \gamma^2 \rho. \tag{208}$$

Note that calculation details are left in Exercise. By using the form of the 4-velocity as

$$U^0 = \frac{dt}{d\tau} = \gamma, \quad U^i = \frac{dx^i}{d\tau} = \gamma v^i. \tag{209}$$

The components of the EMT above can be written as

$$T^{\mu\nu} = (\rho + p) U^\mu U^\nu + p\eta^{\mu\nu}. \tag{210}$$

Now, we have already the form of the perfect fluid which can be expressed in any frame. However, this form can be used only in the flat spacetime. We need to express it in more general where the curved spacetime includes. This can be generalized by replacing $\eta^{\mu\nu}$ with the general metric tensor $g^{\mu\nu}$ and remembering that the indices can rise or lower by using $g^{\mu\nu}$. As a result, the generally covariant form of the EMT for the perfect fluid can be expressed as

$$T^{\mu\nu} = (\rho + p)U^{\mu}U^{\nu} + pg^{\mu\nu}. \tag{211}$$

2.4.2 Friedmann equation dynamics of the universe

We have already known the general form of both metric tensor and EMT. Now, we will perform calculation to find the exact solutions of them. In order to solve for the solution, let us find the component of the Einstein tensor first. By using the FLRW metric in Eq. (203) and the definition of the connection in Eq. (103), each components of the connection can be calculated such as

$$\begin{aligned}
\Gamma^0_{ij} &= \frac{1}{2}g^{00}\left(\cancel{\partial_i g_{0j}} + \cancel{\partial_j g_{0i}} - \partial_0 g_{ij}\right), && (212)\\
&= \frac{1}{2}(-1)\left(-\partial_t(a^2\gamma_{ij})\right), && (213)\\
&= \gamma_{ij}a\dot{a} = \frac{\dot{a}}{a}a^2\gamma_{ij} = Hg_{ij}, && (214)
\end{aligned}$$

where dot denote the derivative with respect to the coordinate t and $H = \dot{a}/a$ is the Hubble parameter. Other non-zero components can be calculated in the same way and then leave for the student to perform it for Exercise. The results can be expressed as

$$\begin{aligned}
&\Gamma^i_{0j} = H\delta^i_j, \quad \Gamma^1_{11} = \frac{\kappa r}{1-\kappa r^2}, \quad \Gamma^1_{22} = -r(1-\kappa r^2), \quad \Gamma^1_{33} = \sin^2\theta\,\Gamma^1_{22},\\
&\Gamma^2_{33} = -\sin\theta\,\cos\theta, \quad \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta. \qquad (215)
\end{aligned}$$

By using these component of the connection and the definition of the Ricci tensor, the non-zero components of the Ricci tensor and the Ricci scalar can be written as

$$\begin{aligned}
R_{00} &= -3(\dot{H} + H^2), && (216)\\
R_{ij} &= \left(\dot{H} + 3H^2 + \frac{2\kappa}{a^2}\right)g_{ij}, && (217)\\
R = g^{\mu\nu}R_{\mu\nu} &= g^{00}R_{00} + g^{ij}R_{ij} = 6\left(\dot{H} + 2H^2 + \frac{\kappa}{a^2}\right). && (218)
\end{aligned}$$

Consequently, the non-zero components of the Einstein tensor can be written as

$$G^0_0 = -3\left(H^2 + \frac{\kappa}{a^2}\right), \tag{219}$$

$$G^i_j = -\left(2\dot{H} + 3H^2 + \frac{\kappa}{a^2}\right)\delta^i_j. \tag{220}$$

The non-zero components of EMT can be expressed as $T^0_0 = -\rho, T^i_j = p\delta^i_j$. Now we have all components of ones in both side of the Einstein equation. By putting these into the Einstein equation, one obtains $(0,0)$ and (i,j) components respectively as

$$3\left(H^2 + \frac{\kappa}{a^2}\right) = 8\pi\, G\, \rho, \tag{221}$$

$$-\left(2\dot{H} + 3H^2 + \frac{\kappa}{a^2}\right) = 8\pi\, G\, p. \tag{222}$$

Now, we have two equations and three variables, a, ρ, p. It might not be possible to solve the exact solutions for them. However, we actually have one more equation, coming from the conservation of the EMT as follows

$$\begin{aligned}
\nabla_\mu T^\mu_\nu &= 0\partial_\mu T^\mu_\nu + \Gamma^\mu{}_{\mu\rho} T^\rho_\nu - \Gamma^\rho{}_{\mu\nu} T^\mu_\rho = 0,\\
\nabla_\mu T^\mu_0 &= \partial_\mu T^\mu_0 + \Gamma^\mu{}_{\mu\rho} T^\rho_0 - \Gamma^\rho{}_{\mu 0} T^\mu_\rho,\\
&= \partial_0 T^0_0 + \Gamma^i{}_{i0} T^0_0 - \Gamma^i{}_{j0} T^j_i,\\
&= -\dot{\rho} - 3H\rho - H\delta^i_j p\, \delta^j_i,\\
&= -(\dot{\rho} + 3H(\rho + p)) = 0,\\
\dot{\rho} + 3H(\rho + p) &= 0.
\end{aligned} \tag{223}$$

Now we have 3 equations and 3 variables to solve. It seem like we can use these equation to completely solve for the solutions. However, one found that only two of them are independent. Actually, one can show that $\partial_t(221) + 3H((\ 221) + (222)) = 0$ gives Eq. (223). In order to solve the exact solution of the equations, one has to impose one more condition. It is convenient to impose the condition between ρ and p to characterize the properties of the perfect fluid. The well-known condition is the "equation of state",

$$p = w\rho, \tag{224}$$

where w is the equation of state parameter. Substituting this into continuity equation Eq. (223), and then solving ρ in terms of a, one obtains

$$\rho = \rho_0 a^{-3(1+w)}, \tag{225}$$

where ρ_0 is the integration constant representing the energy density at the present time at $a = 1$. Now let us consider the universe filling with one of well-known matter/energy

- Non-relativistic matter

Non-relativistic matter is also known in other names such as dust or matter. As we have known dust is the fluid with pressureless. Therefore, the equation of state parameter vanishes, $w = 0$. Substituting $w = 0$ into Eq. (225), we have $\rho \propto a^{-3}$. This is also make our sense since the number of particle is conserved then the energy density is scaled by its volume which is proportional to a^3.

- Relativistic matter

Relativistic matter is also known radiation. From Exercise we found that the energy momentum of radiation (gauge field) is traceless. This corresponds to $\rho + 3p = 0, \rightarrow\ w = 1/3$. Substituting $w = 1/3$ into Eq. (225), we have $\rho \propto a^{-4}$. This is also make our sense since the energy of the photon (or radiation) is also inversely proportional to wavelength which is scaled by a. Together with volume scaling, then the energy density is proportional to a^{-4}.

Now we can go further to find the exact solution. Let substituting $\rho = \rho_0 a^{-3(1+w)}$ into Eq. (221) and then we have

$$3\left(H^2 + \frac{\kappa}{a^2}\right) \ = \ 8\pi\, G\, \rho_0 a^{-3(1+w)}. \tag{226}$$

For simplicity, let us solve this equation in the case of flat universe $k = 0$. As a result, the solution can be written as

$$a = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}, \quad t_0 = \sqrt{12\pi G}(1+w) \tag{227}$$

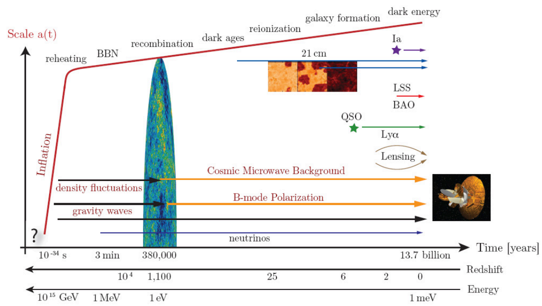
As a result, for matter and radiation we have

- matter: $a \propto t^{2/3}$.
- radiation: $a \propto t^{1/2}$.

Before we finish this section, let consider one of an important equation called acceleration equation. Eliminating κ term in Eq. (221) and Eq. (222), one obtains

$$2\dot{H} + 2H^2 = -\frac{8\pi G}{3}(1+3w)\rho, \quad \rightarrow \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(1+3w)\rho. \tag{228}$$

From this equation, one finds that for the well-known matter/energy (all non-relativistic, $w = 0$ and relativistic, $w = 1/3$ ones), it is not possible to provide the universe with accelerated expansion. Note that this is possible for cosmological constant and leave for students in Exercise. Finally, I will briefly review the standard picture of the universe including the evolution of the universe and the major events in the history of the universe. This is illustrated roughly in Fig. 17 and I will explain more detail in the classroom.



	Time	Energy	
Planck Epoch?	$< 10^{-43}$ s	10^{18} GeV	
String Scale?	$\gtrsim 10^{-43}$ s	$\lesssim 10^{18}$ GeV	
Grand Unification?	$\sim 10^{-36}$ s	10^{15} GeV	
Inflation?	$\gtrsim 10^{-34}$ s	$\lesssim 10^{15}$ GeV	
SUSY Breaking?	$< 10^{-10}$ s	> 1 TeV	
Baryogenesis?	$< 10^{-10}$ s	> 1 TeV	
Electroweak Unification	10^{-10} s	1 TeV	
Quark-Hadron Transition	10^{-4} s	10^{2} MeV	
Nucleon Freeze-Out	0.01 s	10 MeV	
Neutrino Decoupling	1 s	1 MeV	
BBN	3 min	0.1 MeV	
			Redshift
Matter-Radiation Equality	10^{4} yrs	1 eV	10^{4}
Recombination	10^{5} yrs	0.1 eV	1,100
Dark Ages	$10^{5} - 10^{8}$ yrs		> 25
Reionization	10^{8} yrs		$25 - 6$
Galaxy Formation	$\sim 6 \times 10^{8}$ yrs		~ 10
Dark Energy	$\sim 10^{9}$ yrs		~ 2
Solar System	8×10^{9} yrs		0.5
Albert Einstein born	14×10^{9} yrs	1 meV	0

Figure 17: The left panel shows the standard evolution of the universe. The right panel shows the major events in the history of the universe [?].

3 Cosmological model

3.1 Cosmological constant

- The simplest candidate for explain the accelerated expansion of the universe nowadays is known as ΛCDM where the Λ stands for the dark energy contributed from cosmological constant and CDM stands for the contribution of dark matter namely Cold dark matter. This two contribution is about 95% fo the conten in our universe while the contribution from dark energy is about 70% and the contribution from dark matter is about 25%.

- The cosmological constant is the promising candidate for dark energy to drive the accelerated expansion of the universe nowadays

The action for such the model can be expressed as

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R - 2\Lambda) + S_m, \tag{229}$$

where S_m is the action for the matter. Varying the action with respect to the metric tensor, one obtains

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{230}$$

By using the FLRW metric, the Friedmann equation and the acceleration equation can be written as

$$\begin{aligned} \left(H^2 + \frac{\kappa}{a^2}\right) &= \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}, & (231) \\ \frac{\ddot{a}}{a} &= \frac{8\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. & (232) \end{aligned}$$

- When Einstein constructed the Einstein equation, he believed that the universe is static. Therefore, he put the cosmological constant into the Einstein equation to make the universe static.

To achieve such goal, one can find the solution such that $\dot{a} = 0 = \ddot{a}$. Let us examine how the cosmological can be. Firstly let consider the universe filled by dust so that the pressure vanishes. From Eq. (232) and Eq. (231), one obtains

$$\begin{aligned} \frac{8\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} &= 0, \Rightarrow \rho = \frac{\Lambda}{4\pi}, & (233) \\ \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} &= \frac{\kappa}{a^2} \Rightarrow \frac{\kappa}{a^2} = \Lambda. & (234) \end{aligned}$$

- After that, Lemaitre showed that even this is a solution to the equations of motion, it is not stable due to the small perturbation.
- In 1917, de Sitter found that ther exist the solution in the empty space with $H = \sqrt{\Lambda/3}$.
- During 1910-1920, Slipler observed the spectral of the galaxies and found that it is red-shifted.
- In 1922, Friedmann found the evolving solution with the expanding universe.
- In 1927, Lermaitre proposed the "hot Big Bang" model of the universe. This is may be the first model to explain how the universe evolves associated with general relativity. In such the model, the evolution of the universe can be devided into 3 states as follows

1. Fundametal elements are formed and the universe expands from the point source. It dominated by pressureless matter, $a \propto t^n (0 < n < 1)$.

2. Nebulars and galaxies are formed. The phase is represented by the static universe proposed by Einstein, $a \propto const$

3. A period of a fast expansion of the universe, $a \propto t^n (1 < n)$. This phase is realized by de Sitter solution.

- In 1929, Hubble formulated Hubble's law by combining the results of Slipher. The existence of cosmological constant was clearly not required to give rise to the cosmic expansion of the universe.

- In 1945, in the book "The Meaning of Relativity" written by Einstein, he said that "if the Hubble's expansion had been discovered at the time of creation of the general theory of relativity, the cosmological constant would not never have been introduced".

- In 1970, Gamov recalled that "when I was discussuing cosmological problem with Einstein, he remarked that the introduction of the cosmological constant termwas the biggest blunder he ever made in his life". Remarkably, the cosmological constant becomes the main ingredient in the standard cosmology nowadays.

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