# Essence of cosmological perturbation theory

Khamphee Karwan

The Institute for Fundamental Study
Cosmology with Computations Workshop, 2023

# Perturbations in the universe

- According to cosmological principle, the universe is homogeneous and isotropic on large scales.
- In principle, the universe cannot be perfectly homogeneous andisotropic, because structures in the universe such as clusters of galaxy, galaxies, etc, cannot be created in the completely smooth universe.
- In the standard notion, structures in the universe are developed from small inhomogeneity and anisotropy in the early universe.
- The small inhomogeneity and anisotropy can be treated as perturbations around the homogeneous and isotropic universe.

# The Friedmann universe

- Based on observations, the universe is spatially flat, so we will consider only the spatially flat universe in this lecture.
- The spacetime of the homogeneous and isotropic universe is described by FLRW metric, which for the spatially flat universe, is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = a^2 \left( -d\tau^2 + \delta_{ij} dx^i dx^j \right) . \quad (1)$$

  where $\delta_{ij}$ is the kronecker delta, $a$ is a cosmic scale facter, and $d\tau = dt/a$ is a conformal time while $t$ is a time.
- In this lecture, we use the Greek indices running over 0, 1, 2, 3 to represent spacetime components, while we use the Latin indices running over 1, 2, 3 to represent the spatial components.

# The Friedmann universe

- Matter and energy in the homogeneous and isotropic universe can be described by a perfect fluid which the energy-momentum tensor takes the form

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 \\ 0 & g_{ij}P \end{pmatrix} , \qquad (2)$$

where $\rho$ and $P$ are the energy density and pressure of the fluid.

- The component $0-0$ of the Einstein equation yields the Friedmann equation:

$$\mathcal{H}^2 = \frac{1}{3m_p^2}\rho a^2 , \qquad (3)$$

where $\mathcal{H} \equiv \dot{a}/a$ is the Huble parameter, a dot denotes derivative with respect to the conformal time $\tau$, and $\rho$ is the total energy density in the universe.

# The Friedmann universe

- The components $i - i$ of the Einstein equation give the acceleration equation:

$$2\frac{1}{a}\frac{d^2a}{dt^2} + \frac{\mathcal{H}^2}{a^2} = -\frac{1}{m_p^2}P\,, \tag{4}$$

  where $P$ is total pressure.
- The homogeneous and isotropic universe which its dynamics are described by the Friedmann equation and the above equation is the Friedmann universe.
- Dynamics of $\rho$ and $P$ are governed by the conservation of energy-momentum tensor:

$$\nabla_\nu T^{\nu\mu} = 0\,. \tag{5}$$

# The Friedmann universe

- For the Friedmann universe, the conservation law yields

$$\dot{\rho} = -3\mathcal{H}(\rho + P)\,. \qquad (6)$$

- For the Friedmann universe, we have the Friedmann equation, the acceleration equation, and one conservation equation.
- Nevertheless, the acceleration equation can be obtained by differentiating the Friedmann equation with respect to time, and using the conservation equation to eliminate $\dot{\rho}$.
- Hence, we have two independent evolution equations, but we have three time-dependent variables, i.e., $a, \rho$ and $P$

# The Friedmann universe

- Dynamics of the Friedmann universe can be completely specified if we know the relation between $\rho$ and $P$, which for the perfect fluid is given vy

$$P = w\rho\,. \tag{7}$$

- this relation is the equation of state and $w$ is the equation of state parameter.
- The equation of state parameter of radiation, matter and cosmological constant are $1/3$, 0 and -1.

# Perturbation in the metric tensor

- In the subsequent topics, we will study how to quantify perturbations in spacetime and matter in the universe by decomposing metric and energy-momentum tensors into background and perturbed parts.
- The reference for the subsequent topics is [arXiv:astro-ph/0101563].

# Perturbation in the metric tensor

- To descripe small deviation from the homogeneity and isotropy of the spacetime, we decompose the metric tensor into background part and perturbed part as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \,, \tag{8}$$

  where $\bar{g}_{\mu\nu}$ and $h_{\mu\nu}$ are tthe background and perturbed metrics
- We will use an over bar to denote the homogeneous and isotropic background quantities.
- The metric $\bar{g}_{\mu\nu}$ is the FLRW metric, while $|h_{\mu\nu}| < 1$.
- The components of the metric can be determined from symmetry of the systems.

# Perturbation in the metric tensor

- In the perturbed universe, there is no special symmetry to determine the components of the perturged metric.
- Hence, we parameterize each components of the perturbed metric as follows:
- The component $0-0$ can be espressed in term of a scalar function $\psi(\tau,\vec{x})$ as

$$g_{00} = -a^2(\tau)\left(1+2\psi(\tau,\vec{x})\right) . \tag{9}$$

# Perturbation in the metric tensor

- The components $0 - i$ and $i - 0$ can be espressed in term of a three-dimensional vector.

$$g_{i0} = g_{0i} = a^2(\tau) v_i(\tau, \vec{x}) \, . \tag{10}$$

- From vector analysis, any vector field can be decomposed into curl- and divergence-free parts, so that

$$g_{i0} = g_{0i} = a^2(\tau) \left( B_{,i}(\tau, \vec{x}) - S_j(\tau, \vec{x}) \right) \, , \tag{11}$$

  where subscript $_{,i}$ denotes $\partial / \partial x^i$ and $S_{i,i} = 0$.

- We see that $g_{i0}$ can be expressed in terms of a scalar function and divergence-free vector.

# Perturbation in the metric tensor

- The component $h_{ij}$ can be expressed in terms of scalar function as

$$h_{ij} = a^2(\tau)\left(-2\phi(\tau,\vec{x})\delta_{ij} + 2E_{,i,j}(\tau,\vec{x})\right) . \qquad (12)$$

- This component of the metric can also be expressed in terms of the divergence-free vector $F_i$ as

$$h_{ij} = a^2(\tau)\left(F_{i,j}(\tau,\vec{x}) + F_{j,i}(\tau,\vec{x})\right) . \qquad (13)$$

- The last part of $h_{ij}$ is the three-dimensional tensor which is tressless and divergence-free :

$$h_{ij} = a^2(\tau)H_{ij}(\tau,\vec{x})\,, \quad \text{where} \quad H^i_i = H^{ij}_{,j} = 0\,. \qquad (14)$$

# Perturbation in the metric tensor

- Hence, we have

$$g_{ij} = a^2 \left[ (1 - 2\phi)\delta_{ij} + 2E_{,i,j} + F_{i,j} + F_{j,i} + H_{ij} \right] . \tag{15}$$

- The perturbed metric can be parameterized by four scalar functions, two divergence-free vectors, and a traceless and divergence-free tensor.
- There are four degrees of freedom from four scalar functions, four degrees of freedom from two divergence-free vectors, and two degrees of freedom from tressless and divergence-free tensor, so that we have ten degrees of freedom in total.
- These scalar, 3-D vector and 3-D tensor fields completely characterized components of the metric tensor.

# Perturbation in the metric tensor

- For linear perturbations, these scalar, vector and tensor fields evolve independently.
- This means that the perturbation in metric tensor can be decomposed into scalar, vector and tensor perturbations (or modes).

# Perturbation in the energy-momentum tensor

- The energy-momentum tensor of the perfect fluid can be written in the general form as

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + g_{\mu\nu} P \,, \tag{16}$$

where $\rho$, $P$ and $u^\mu$ are the energy density, pressure, and four-velocity of the fluid.

- The energy density and pressure are scalar quantities, they can be decomposed into a background and a perturbed parts as

$$\rho = \bar{\rho}(\tau) + \delta\rho(\tau, \vec{x}) \,, \quad \text{and} \quad P = \bar{P}(\tau) + \delta P(\tau, \vec{x}) \,. \tag{17}$$

- For the homogeneous and isotropic background, the spatial component of the four-vector must vanish, i.e.,

$$u^\mu = \frac{1}{a}(1, 0, 0, 0) \,, \quad u_\mu = a(-1, 0, 0, 0) \,, \tag{18}$$

# Perturbation in the energy-momentum tensor

- The temporal and spatial components of the four-velocity can also be expressed in terms of scalar functions and divergence-free vector $V^i$, so that

$$u^\mu = \frac{1}{a}\left(u^0, v^{,i} + V^i\right) \,, \quad u_\mu = a\left(u_0, v_{,i} + V_i\right) \,, \qquad (19)$$

- The temperal component of $u^\mu$ is estimated from

$$g_{\mu\nu} u^\mu u^\nu = -1 \,, \quad \rightarrow \quad u^0 = \frac{1}{a}(1-\psi) + \text{higher order} \,. \qquad (20)$$

- Hence, we have

$$u^\mu = \frac{1}{a}\left(1-\psi, v^{,i} + V^i\right) \,, \quad u_\mu = a\left(-1-\psi, v_{,i} + V_i\right) \,, \qquad (21)$$

# Perturbation in the energy-momentum tensor

- The energy-momentum tensor can be expressed up to the first order perturbations as

$$
\begin{aligned}
T_0^0 &= -(\bar{\rho} + \delta\rho) , && (22)\\
T_i^0 &= (\bar{\rho} + \bar{P})(B_{,i} + v_{,i} + V_i - S_i) , && (23)\\
T_0^i &= -(\bar{\rho} + \bar{P})(v^{,i} + V^i) , && (24)\\
T_j^i &= (\bar{P} + \delta P)\,\delta_j^i . && (25)
\end{aligned}
$$

# Perturbation in the energy-momentum tensor

- For general fluid, the component i-j of energy-momentum tensor can contain traceless and divergence-free part:

$$T^i_j = \left(\bar{P} + \delta P\right) \delta^i_j + \pi^i_j \,, \tag{26}$$

where $\pi^i_j$ describes anisotropy in the spatial part of the energy-momentum tensor.

- The anisotropic perturbation $\pi^i_j$ can be decomposed into scalar, vector and tensor parts as

$$\pi^i_j = \Pi_{,j}^{;i} - \frac{1}{3}\Delta^2 \Pi \delta^i_j + \frac{1}{2}\left(\pi^i_{,j} + \pi_j^{;i}\right) + \Pi^i_j \,, \tag{27}$$

where $\Delta^2 \equiv \partial_i \partial^i$, $\pi^i_{,i} = \Pi^i_i = \Pi^i_{j,i} = 0$.

# Gauge degrees of freedom

- Inserting the perturbed metric and perturbed energy-momentum tensors into the Einstein equation, we get two evolution equations and two constraint equations as follows:
- Components $0-0$ and $0-i$ yield energy and momentum constraint equations

$$3\mathcal{H}\left(\dot{\phi}+\mathcal{H}\psi\right)-\Delta^2\phi-\mathcal{H}\Delta^2\sigma=-\frac{m_p^2}{2}a^2\delta\rho\,, \tag{28}$$

$$\dot{\phi}+\mathcal{H}\psi=-\frac{m_p^2}{2}a^2\left(\bar{\rho}+\bar{P}\right)(v+B)\,, \tag{29}$$

where $\sigma_s\equiv-B+\dot{E}$ is the shear perturbation.

# Gauge degrees of freedom

- Components $i-i$ and $i \neq j$ yield two evolution equations

$$\dot{\phi} + 2\mathcal{H}\dot{\phi} + \mathcal{H}\dot{\psi} + \left(2\dot{\mathcal{H}} + \mathcal{H}^2\right)\psi = \frac{m_p^2}{2}a^2\delta P \,, \quad (30)$$

$$\dot{\sigma}_s + 2\mathcal{H}\sigma_s - \psi + \phi = m_p^2 a^2 \Pi \,, \quad (31)$$

- The temporal and spatial components of $\nabla_\nu T_\mu^\nu = 0$ give the conservation equations for energy and momentum

$$\delta\dot{\rho} + 3\mathcal{H}(\delta\rho + \delta P) = (\bar{\rho} + \bar{P})\left[3\dot{\phi} - \Delta^2(v + \dot{E})\right] \,, \quad (32)$$

$$\frac{\partial}{\partial\tau}\left[(\bar{\rho} + \bar{P})(v + B)\right] + \delta p + \frac{2}{3}\Delta^2\Pi = -(\bar{\rho} + \bar{P})\left[\psi + 4\mathcal{H}(v + B)\right] \,. \quad (33)$$

# Gauge degrees of freedom

- For scalar perturbation, the metric perturvation can be described by four fields, $\phi, B, E$ and $\psi$.
- The perturbation in energy-momentum tensor can also be quantified by four fields, $\delta\rho, \delta P, v$ and $\Pi$.
- However, if we combine the evolution equations from $\nabla\nu T^{\nu}_{\mu} = 0$, and the perturbed Einstein equations, only five equations are linearly independent.
- In general, the relation between $\delta\rho$ and $\delta P$ is given by

$$\delta P = c_s^2 \delta\rho \,, \tag{34}$$

where $c_s^2$ is the sound speed square of the perturbations.

# Gauge degrees of freedom

- This suggests that two degrees of freedom are not physical degrees of freedom. They ar gauge degrees of freedom.
- The gauge degrees of freedom are the consequences of diffeomorphism invariance of GR.
- The gauge degrees of freedom can be eliminated by performing calculations in suitable hypersurfaces or by using gauge-invariant variables which are suitable combination of perturbed variables.
- In this lecture, we briefly discuss how to eliminate gauge degrees of freedom by choosing suitable hypersurfaces, and we will focus on scalar perturbations.

# Gauge degrees of freedom

- The gauge transformations in GR are the coordinate transformations.
- The infinitesimal coordinate transformations can be written as

$$\tilde{\tau} = \tau + \xi^0(\tau, x^i)\,, \quad \tilde{x}^i = x^i + \xi^{,i}(\tau, x^i)\,, \tag{35}$$

where $\xi^0$ and $\xi$ are gauge degrees of freedom,

# Gauge degrees of freedom

- The metric tensor can be written in terms of the line element $ds^2$ which is a scalar quantity as

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= a^2(\tau)\Big\{ -(1+2\psi)d\tau^2 + 2B_{,i} d\tau dx^i \\ &\quad + \left[(1-2\phi)\delta_{ij} + 2E_{,ij}\right] dx^i dx^j \Big\} . \end{aligned} \tag{36}$$

- We now consider gauge transformation of this line element.
- Up to the first order in the coordinate transformations, the infinitesimal coordinate transformations yield

$$\xi^0(\tau, x^i) = \xi^0(\tilde{\tau}, \tilde{x}^i) , \quad \xi(\tau, x^i) = \xi(\tilde{\tau}, \tilde{x}^i) , \tag{37}$$

# Gauge degrees of freedom

- The inverse coordinate transformations are

$$\tau = \tilde{\tau} - \xi^0(\tilde{\tau}, \tilde{x}^i)\,, \quad x^i = \tilde{x}^i - \xi^{,i}(\tilde{\tau}, \tilde{x}^i)\,, \tag{38}$$

- Hence, we have

$$d\tau = d\tilde{\tau} - \dot{\xi}^0 d\tilde{\tau} - \xi^0_{,i} d\tilde{x}^i\,, \quad dx^i = d\tilde{x}^i - \dot{\xi}^{,i} d\tilde{\tau} - \xi^{,i}_{,j} d\tilde{x}^j\,. \tag{39}$$

- For the scale factor, we have

$$a(\tau) = a(\tilde{\tau}) - \xi^0 \dot{a}(\tilde{\tau})\,. \tag{40}$$

# Gauge degrees of freedom

- Each parts of The line element in new coordinates is transformed as

$$\begin{aligned} & -a^2(\tau)(1+2\psi)d\tau^2 \\ = & -a^2(1-2\xi^0\mathcal{H})(1+2\psi)(d\tilde{\tau}^2 - 2\dot{\xi}^0 d\tilde{\tau}^2 \\ & -2\xi^0_{,i} d\tilde{x}^i d\tilde{\tau}) \,. \end{aligned} \tag{41}$$

$$\begin{aligned} & 2a^2(\tau)B_{,i}d\tau dx^i \\ = & 2a^2 B_{,i} d\tilde{\tau} d\tilde{x}^i + \text{higher order} \,. \end{aligned} \tag{42}$$

$$\begin{aligned} & a^2(1-2\xi^0\mathcal{H})\left[(1-2\phi)\delta_{ij} + 2E_{,ij}\right] dx^i dx^j \\ = & a^2(1-2\xi^0\mathcal{H})\left[(1-2\phi)\delta_{ij} + 2E_{,ij}\right] \times \\ & (d\tilde{x}^i d\tilde{x}^j - 2\dot{\xi}^{,i} d\tilde{\tau} d\tilde{x}^j - 2\xi^{,i}_{,k} d\tilde{x}^k d\tilde{x}^j) \,. \end{aligned} \tag{43}$$

# Gauge degrees of freedom

- Combining all parts, we get

$$
\begin{aligned}
ds^2 &= -a^2\left[1+2(\underbrace{\psi-\xi^0\mathcal{H}-\dot{\xi}^0}_{=\tilde{\psi}})\right]d\tilde{\tau}^2 \\
&\quad +2a^2(\underbrace{B_{,i}+\xi^0_{,i}-\dot{\xi}_{,i}}_{=\tilde{B}_{,i}})d\tilde{\tau}d\tilde{x}^i \\
&\quad +a^2\left[\left(1-2(\underbrace{\phi+\xi^0\mathcal{H}}_{=\tilde{\phi}})\delta_{ij}\right)+2(\underbrace{E_{,ij}-\xi_{,ij}}_{=\tilde{E}_{,ij}})\right]d\tilde{x}^i d\tilde{x}^j
\end{aligned}
\tag{44}
$$

# Gauge degrees of freedom

- Under gauge transformations, the perturbation variables are transformed as

$$\tilde{\psi} = \psi - \mathcal{H}\xi^0 - \dot{\xi}^0 , \quad \tilde{B} = B + \xi^0 - \dot{\xi} , \tag{45}$$

$$\tilde{\phi} = \phi + \mathcal{H}\xi^0 , \quad \tilde{E} = E - \xi . \tag{46}$$

- The spatial gauge degree of freedom can be fixed If we choose the gauge transformation such that $\xi = E$ which yields $\tilde{E} = 0$.
- The temporal gauge degree of freedom can be fixed if we choose $\xi^0 = -B + \dot{E}$.

# Gauge degrees of freedom

- This means that if we work in the hypersurface on which $\tilde{B} = \tilde{E} = 0$, both temporal and spatial gauge degrees of freedom are completely fixed.
- The hypersurface on which $\tilde{B} = \tilde{E} = 0$ can be reached from any hypersurfaces if we choose $\xi = E$ and $\xi^0 = \dot{E} - B$.
- This gauge choice is the Conformal Newtonian Gauge.
- We see that two gauge degrees of freedom could be eliminated/fixed if we set two of the perturbation variables to zero.

# Gauge degrees of freedom

- Fixing gauge can also be done by choosing the gauge degrees of freedom based on the gauge transformation properties of the perturbations in a fluid.
- In general, the gauge degrees of freedom may be not completely fixed even though two perturbation variables are set to zero.
- One of the gauge choices, which has been used in literature, is the Synchronous gauge in which the perturbations in the temporal components of the metric vanish, i.e., $\tilde{\psi} = \tilde{B} = 0$.
- For the Synchronous gauge, we have

$$\xi^0 = \frac{1}{a} \int d\tau a\psi + \frac{f(x^i)}{a}, \qquad (47)$$

where $f(x^i)$ is an arbitrary function of spatial coordinates which is a residual gauge freedom.

# Gauge degrees of freedom

- To fix a residual gauge freedom, we have to put additional conditions such as symmetry of the system or work in special frame of fluid, e.g., rest frame of some fluid component in the universe.
- In the subsequent topics, we will present the evolution equations for the linear perturbations in the universe.
- The reference for these topics is [arXiv:astro-ph/9506072].

# Evolution equations for perturbations

- In the following studies, we will focus on the Newtonian gauge in which the line element takes the form

$$ds^2 = a^2(\tau)\left\{-(1+2\psi)d\tau^2 + (1-2\phi)dx^i dx_i\right\} . \quad (48)$$

- The following equations will be expressed in the Fourier space

$$\psi(\tau, \vec{x}) = \int d^3k e^{i\vec{k}\cdot\vec{x}}\psi(\tau, \vec{k}) \quad (49)$$

- According to our reference, velocity perturbation is described in terms of its divergence, and the anisottropic perturbation is described in terms of shear stress:

$$(\bar{\rho}+\bar{P})\theta \equiv ik^j \delta T^0{}_j\,, \quad (\bar{\rho}+\bar{P})\sigma \equiv -(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij})\Sigma^i{}_j\,, \quad (50)$$

where $\hat{k} = \vec{k}/|\vec{k}|$.

# Evolution equations for perturbations

- Here, the traceless component of $T^i{}_j$ is

$$\Sigma^i{}_j \equiv T^i{}_j - \frac{1}{3}\delta^i{}_j T^k{}_k\,, \tag{51}$$

- The perturbed Einstein equations in the Newtonian gauge yield

$$3\mathcal{H}\left(\dot{\phi} + \mathcal{H}\psi\right) + k^2\phi = -\frac{m_p^2}{2}a^2\delta\delta\rho\,, \tag{52}$$

$$k^2\left(\dot{\phi} + \mathcal{H}\psi\right) = \frac{m_p^2}{2}a^2(\bar{\rho} + \bar{P})\theta\,, \tag{53}$$

$$\ddot{\phi} + \mathcal{H}(\dot{\psi} + 2\dot{\phi}) + \left(2\dot{\mathcal{H}} + \mathcal{H}^2\right)\psi$$

$$+\frac{k^2}{3}(\phi - \psi) = \frac{m_p^2}{2}a^2\delta P\,, \tag{54}$$

$$k^2(\phi - \psi) = \frac{3m_p^2}{2}a^2(\bar{\rho} + \bar{P})\sigma\,, \tag{55}$$

# Evolution equations for perturbations

- The perturbations in energy density $\delta\rho$ is described by the dimensionless density contrast $\delta \equiv \delta\rho/\bar{\rho}$.
- The cold dark matter (CDM) can be described by a perfect fluid, so that the evolution equations for its perturbations obey $\nabla_\nu T^\nu_\mu = 0$:

$$\dot{\delta}_c = -\theta_c + 3\dot{\phi}\,, \quad \dot{\theta}_c = -\frac{\dot{a}}{a}\theta_c + k^2\psi\,, \tag{56}$$

  where subscript $_c$ denotes CDN.
- During some epoch, the mean free path of photons and neutrinos are very long, so that they perturbations cannot be described by perfect fluid.

# Evolution equations for perturbations in radiation

- To describe perturbations in photon and neutrino, we use the distribution function which is defined on the phase space.
- The phase space is spanned by conjugate momentum and coordinates.
- The conjugate momentum is the spatial component of the 4-momentum with lower indices denoted by $P_i$.
- To remove the contribution from the metric tensor, it is convenien to work with momentum in the orthogonal bases $p_i$ defined as

$$\eta^{\mu\nu} p_\mu p_\nu = -m^2 = g^{\mu\nu} P_\mu P_\nu \,. \tag{57}$$

Here, $\eta^{\mu\nu}$ is the Minkowski metric and $m$ is a rest mass of a particle.

# Evolution equations for perturbations in radiation

- The relation between $P_\mu$ and $p_\mu$ is given by

$$P_\mu = \mathrm{e}_\mu^\alpha p_\alpha\,, \quad \text{so that} \quad g^{\mu\nu}\mathrm{e}_\mu^\alpha \mathrm{e}_\nu^\beta = \eta^{\alpha\beta}\,. \tag{58}$$

- For the newtonian gauge, we get

$$P_0 = (1+\psi)p_0\,, \quad P_i = \mathrm{e}_i^j p_j = a(1-\phi)p_i\,. \tag{59}$$

- The energy of a particle defined in orthogonal bases is

$$\epsilon = \sqrt{p_i p^i + m^2} = \sqrt{p^2 + m^2}\,, \quad \rightarrow \quad P_0 = -(1+\psi)\epsilon\,. \tag{60}$$

# Evolution equations for perturbations in radiation

- An infinitesimal volume of phase space is $dV = dx^1 dx^2 dx^3 dP_1 dP_2 dP_3$, and number of particles in a unit volume is

$$dN = f(x^i, P_j, \tau) dV \, . \tag{61}$$

- In the background universe, the distribution function describes is the Fermi-Dirac distribution for fermions and the Bose-Einstein distribution for bosons given by

$$f_0(\epsilon) = \frac{g_s}{2\pi^2} \frac{1}{\mathrm{e}^{\epsilon/T_0} \pm 1} \, , \tag{62}$$

where $T_0 = aT$ denotes the present temperature of the particles, the factor $g_s$ is the number of spin state.

# Evolution equations for perturbations in radiation

- The energy-momentum tensor can be written in terms of the distribution function and the 4-momentum components as

$$T_{\mu\nu} = \int dP_1 dP_2 dP_3 \, (-g)^{-1/2} \, \frac{P_\mu P_\nu}{P^0} f(x^i, P_j, \tau) \, , \quad (63)$$

- In the perturbed universe, it is convenien to quantify the momentum in terms of

$$q_j \equiv a p_j \, , \quad (64)$$

- Moreover, we define the direction of the momentum $q_i$ through the unit vector $n_i$ by

$$q_j = q n_j \, , \quad \text{where} \quad n^i n_i = \delta_{ij} = 1 \, . \quad (65)$$

# Evolution equations for perturbations in radiation

- the distribution function can be decomposed into the background and perturbed parts as

$$f(x^i, P_j, \tau) = f_0(q)\left[1 + \Psi(x^i, q, n_j, \tau)\right], \qquad (66)$$

where we have used $\epsilon = q$ for massless particles.

- The perturbations in $(-g)^{-1/2}$ and the volume element are

$$(-g)^{-1/2} = a^{-4}(1-\psi+3\phi), \quad dP_1 dP_2 dP_3 = (1-3\phi)q^2 dq d\Omega, \qquad (67)$$

where $d\Omega$ is the solid angle associated with direction $n_i$.

- For example, the component 0-0 of the energy-momentum tensor is

$$T^0{}_0 = -a^{-4}\int q^3 dq\, d\Omega\, f_0(q)\,(1+\Psi). \qquad (68)$$

# Evolution equations for perturbations in radiation

- For the background, we have

$$\bar{\rho} = a^{-4} 4\pi \int q^3 dq\, f_0(q)\,. \tag{69}$$

- For the perturbation, we get

$$\delta\rho = 3\delta P = a^{-4} \int q^2 dq d\Omega\, q f_0(q) \Psi\,. \tag{70}$$

- In the Fourier space, the distribution function becomes

$$f(k^i, q, n_j, \tau) \rightarrow f(x^i, q, n_j, \tau)\,, \tag{71}$$

where $k^i$ is a wavenumber of the Fourier modes.

# Evolution equations for perturbations in radiation

- We can integrate out $q$ from the distribution function, and expand the angular-dependent part of the resulting function in a series of Legendre polynomials $P_l(\hat{k}\cdot\hat{n})$ as

$$F(\vec{k},\hat{n},\tau) \equiv \frac{\int q^3 dq f_0(q)\Psi}{\int q^3 dq f_0(q)} = \sum_{l=0}^{\infty}(-i)^l(2l+1)F_l(\vec{k},\tau)P_l(\mu)\,, \tag{72}$$

where $\mu \equiv \hat{k}\cdot\hat{n} = \cos\theta$.

- For example:

$$\begin{aligned}
\delta &\equiv \frac{\delta\rho}{\bar{\rho}} = \int d\Omega \frac{\int q^3 dq f_0(q)\Psi}{4\pi\int q^3 dq\, f_0(q)} \\
&= \frac{1}{4\pi}\int d\Omega F(\vec{k},\hat{n},\tau) \\
&= \frac{1}{2}\int d\mu \sum_{l=0}^{\infty}(-i)^l(2l+1)F_l(\vec{k},\tau)P_l(\mu) = F_0\,,
\end{aligned}$$

# Evolution equations for perturbations in radiation

- In the previous calculation, we have used $d\Omega = 2\pi \sin\theta d\theta = -2\pi d\cos\theta = -2\pi d\mu$, and

$$\int_{-1}^{1} P_l(x) P_m(x) dx = \frac{2}{2l+1}\delta_{lm} . \tag{73}$$

- For $\theta$ and $\sigma$, one can show that

$$\theta = \frac{3}{4}kF_1 , \quad \sigma = \frac{1}{2}F_2 . \tag{74}$$

# Evolution equations for perturbations in radiation

- The evolution of the distribution function is described by the Boltzmann equation which is given in real space by

$$\frac{\partial f}{\partial \tau} + \frac{dx^i}{d\tau}\frac{\partial f}{\partial x^i} + \frac{dq}{d\tau}\frac{\partial f}{\partial q} + \frac{dn_i}{d\tau}\frac{\partial f}{\partial n_i} = \left(\frac{\partial f}{\partial \tau}\right)_C , \quad (75)$$

  where the term on the RHS corresponds to the collision between the group of particles described by the distribution function and the other particles.
- The term $dx^i/d\tau$ can be written as

$$\frac{dx^i}{d\tau} = \frac{d\lambda}{d\tau}\frac{dx^i}{d\lambda} = \frac{d\lambda}{d\tau}P^i , \quad (76)$$

  where $\lambda$ is an affine parameter.
- We can compute $d\lambda/d\tau$ as

$$P^\alpha = \frac{dx^\alpha}{d\lambda} , \quad \rightarrow \quad \frac{d\lambda}{d\tau} = \frac{1}{P^0} . \quad (77)$$

# Evolution equations for perturbations in radiation

- Hence,
$$\frac{dx^i}{d\tau} = \frac{P^i}{P^0} = \frac{qn^i}{\epsilon}\,, \tag{78}$$
where we keep the first order in perturbation.
- The term $dq/d\tau$ can be computed from the geodesic equation
$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0\,. \tag{79}$$
- Using $d\lambda/d\tau$ on the previous page, the above equation becomes
$$P^0\frac{dP^\mu}{d\tau} + \Gamma^\mu{}_{\alpha\beta}P^\alpha P^\beta = 0\,, \tag{80}$$

# Evolution equations for perturbations in radiation

- Hence, we get

$$\frac{dq}{d\tau} = q\dot{\phi} - \epsilon(q,\tau)\, n_i \partial_i \psi\,. \tag{81}$$

- The terms $dn_i/d\tau$ and $\partial f/\partial n_i$ are first order in perturbation, so that we ignore the multiplication of these terms.
- The Boltzmann equation in the Fourier space is

$$\frac{\partial \Psi}{\partial \tau} + i\,\frac{q}{\epsilon}\,(\vec{k}\cdot\hat{n})\,\Psi + \frac{d\ln f_0}{d\ln q}\left[\dot{\phi} - i\,\frac{\epsilon}{q}(\vec{k}\cdot\hat{n})\,\psi\right] = \frac{1}{f_0}\left(\frac{\partial f}{\partial \tau}\right)_C\,. \tag{82}$$

# Evolution equations for perturbations in neutrino

- Let us first consider the case of massless neutrino, for this case there is no collision term, so that

$$\frac{\partial \Psi}{\partial \tau} + i\,\frac{q}{\epsilon}\,(\vec{k}\cdot\hat{n})\,\Psi + \frac{d\ln f_0}{d\ln q}\left[\dot{\phi} - i\,\frac{\epsilon}{q}(\vec{k}\cdot\hat{n})\,\psi\right] = 0\,. \quad (83)$$

  In the following calculation, we will set $\epsilon = q$.

- We integrate the above equation over $q^3 dq$ and divide the result by $\int q^3 dq$ as

$$\int q^3 dq \left\{ \frac{\partial f_0 \Psi}{\partial \tau} + ik\mu f_0 \Psi + q\frac{df_0}{dq}\left[\phi' - ik\mu\psi\right] \right\} = 0\,,$$

$$\frac{\partial F}{\partial \tau} + ikP_1(\mu)F + \underbrace{\int q^4 dq \frac{df_0}{dq}}_{=-4\int q^3 dq f_0} \frac{\left[\phi' - ikP_1(\mu)\psi\right]}{\int q^3 f_0 dq} = 0\,,$$

$$\frac{\partial F}{\partial \tau} + ikP_1(\mu)F - 4P_0(\mu)\left[\phi' - ikP_1(\mu)\psi\right] = 0\,. \quad (84)$$

# Evolution equations for perturbations in neutrino

- To extract the evolution equations for the multipole moments, we multiply the equation on the previous slide by $P_l(\mu)$ and integrate the result over $\mu$.
- The coupled differential equations for the multipoles expansion of neutrino distribution:

$$\begin{aligned}
\dot{\delta}_\nu &= -\frac{4}{3}\theta_\nu + 4\dot{\phi}\,, \\
\dot{\theta}_\nu &= k^2\left(\frac{1}{4}\delta_\nu - \sigma_\nu\right) + k^2\psi\,, \\
\dot{F}_{\nu\,l} &= \frac{k}{2l+1}\left[lF_{\nu\,(l-1)} - (l+1)F_{\nu\,(l+1)}\right]\,, \quad l \geq 2\,.
\end{aligned} \tag{85}$$

# Evolution equations for perturbations in photon

- We now consider the Boltzmann equation for the photon.
- For the photon, there is a coupling between photon and charged particles.
- The coupling arisen from the Compton scattering between photons and electrons and the effects of scattering are transfered to baryons through the Coulomb interaction between baryons and electrons.
- Since the mass of the charged baryons (protons) is much larger than the mass of electrons, the main contribution to the scattering process comes from baryons (through baryon velocity).
- Hence, This scattering process is usually called the coupling between photons and baryons.

# Evolution equations for perturbations in photon

- The Boltzmann equation for photon is

$$\frac{\partial F}{\partial \tau} + ik(\mu F - 4\left[\phi' - ik\mu\psi\right]$$
$$= a\sigma_T n_e \left[F_0 + 4ik\mu v_b - F - \frac{1}{2}F_2 P_2\left(\mu\right)\right] . \quad (86)$$

# Evolution equations for perturbations in photon

- The EOM for the multipole moments of photon perturbations are

$$
\begin{aligned}
\dot{\delta}_\gamma &= -\frac{4}{3}\theta_\gamma + 4\dot{\phi}\,, \\
\dot{\theta}_\gamma &= k^2\left(\frac{1}{4}\delta_\gamma - \sigma_\gamma\right) + k^2\psi + \underbrace{an_e\sigma_T(\theta_b - \theta_\gamma)}_{\text{coupling}}, \\
\dot{F}_{\gamma\,2} &= 2\dot{\sigma}_\gamma = \frac{8}{15}\theta_\gamma - \frac{3}{5}kF_{\gamma\,3} \\
&\quad \underbrace{-\frac{9}{5}an_e\sigma_T\sigma_\gamma}_{\text{coupling}} + \frac{1}{10}an_e\sigma_T\left(G_{\gamma\,0} + G_{\gamma\,2}\right)\,, \\
\dot{F}_{\gamma\,l} &= \frac{k}{2l+1}\left[lF_{\gamma\,(l-1)} - (l+1)F_{\gamma\,(l+1)}\right] \\
&\quad - \underbrace{an_e\sigma_T F_{\gamma\,l}}_{\text{coupling}}, \quad l \geq 3 \qquad (87)
\end{aligned}
$$

# Evolution equations for perturbations in baryon

- For baryon, we have to add the contribution from energy-momentum transfer due to photon coupling to the evolution equation for $\theta_b$, such that the photon-baryon fluid is conserved.
- The evolution equations for perturbations in baryon are

$$\dot{\delta}_b = 3\dot{\phi} - \theta_b \,, \qquad (88)$$

$$\dot{\theta}_b = -\mathcal{H}\theta_b + k^2 c_b^2 \delta_b + k^2\psi + \underbrace{\frac{4\bar{\rho}_\gamma}{3\bar{\rho}_b} a n_e \sigma_T (\theta_\gamma - \theta_b)}_{\text{coupling}} \,, \qquad (89)$$

where $c_b^2$ is a baryon sound speed.

# Tight coupling

- Before recombination the electrons number density $n_e$ is large, so that photons and baryons are tightly coupled, with

$$a n_e \sigma_T \equiv \tau_c^{-1} \gg \mathcal{H} \sim \tau^{-1} \,. \tag{90}$$

- In the tight coupling limit, we suppose that $\tau_c/\tau \ll 1$ and $k\tau_c \ll 1$.
- In this limit, we have

$$\dot{\theta}_\gamma = a n_e \sigma_T (\theta_b - \theta_\gamma) \,, \tag{91}$$

$$\dot{\theta}_b = \frac{4\bar{\rho}_\gamma}{3\bar{\rho}_b} a n_e \sigma_T (\theta_\gamma - \theta_b) \,. \tag{92}$$

# Tight coupling

- Hence,

$$\dot{\theta}_\gamma - \dot{\theta}_b = -a n_e \sigma_T \left(1 + \frac{4\bar{\rho}_\gamma}{3\bar{\rho}_b}\right)(\theta_\gamma - \theta_b)\,, \qquad (93)$$

  yielding $\theta_\gamma \to \theta_b$ exponentially.
- Due to tight coupling, we have

$$\dot{F}_{\gamma\, l} = -a n_e \sigma_T F_{\gamma\, l}\,, \quad l \geq 3\,. \qquad (94)$$

  Hence, $F_{\gamma\, l}$ decreases exponentially.

# The acoustic oscillation

- The temperature perturbations is related to the average distribution function $F$ as

$$\Delta_T = \frac{1}{4} F_\gamma . \quad \Rightarrow \quad \Delta_{T0} = \frac{1}{4} F_{\gamma 0} = \frac{1}{4} \delta_\gamma . \qquad (95)$$

- Above the photon diffusion scale, the photons and baryons are tightly coupled.
- At recombination, the diffusion length is much smaller than the Hubble horizon, implying that we consider small scales perturbations.

# The acoustic oscillation

- The evolution equation for the temperature perturbations is

$$\ddot{\Delta}_{T0} + \frac{\dot{R}}{1+R}\dot{\Delta}_{T0} + k^2 c_s^2 \Delta_{T0} = \mathcal{F}\,, \tag{96}$$

  where $c_s^2$ is the sound speed of the photon-baryon fluid defined as

$$c_s^2 \equiv \frac{1}{3}\frac{1}{1+R}\,, \tag{97}$$

  and the force term is

$$\mathcal{F} \equiv \ddot{\phi} - \frac{k^2}{3}\psi + \frac{\dot{R}}{1+R}\dot{\phi}\,. \tag{98}$$

- The above equation describes the oscillation of the photon-baryon fluid in the gravitational driving forces due to external potentials.

# The acoustic oscillation

- The solution can be computed using the Green method as

$$\begin{aligned} & [1+R]^{1/4}\Delta_{T0} \\ = \quad & \Delta_{T0}|_I \cos kr_s + \frac{\sqrt{3}}{k}[\dot{\Delta}_{T0} + \frac{1}{4}\dot{R}\Delta_{T0}]_I \sin kr_s \qquad (99) \\ & + \frac{\sqrt{3}}{k}\int_{\tau_I}^{\tau} d\tau'[1+R(\tau')]^{3/4} \sin[kr_s(\tau) - kr_s(\tau')]F(\tau')\,, \end{aligned}$$

  where a subscript $_I$ denotes evaluation at the initial time.
- Here, $r_s$ is the sound horizon defined as

$$r_s \equiv \int_0^{\tau} c_s d\tau'\,. \qquad (100)$$

# The acoustic oscillation

- For an adiabatic perturbations, we have

$$\Delta_{T0} \propto \sin(kr_s) \tag{101}$$

- Hence, $\Delta_{T0}$ has peaks at

$$kr_s = (n-1)\pi\,, \quad n \in \{1, 2, 3, ...\}\,. \tag{102}$$

# Perturbations on large scales

- On large scales the effect of the tight coupling is negligible, so that the evolution equations for radiation (photon + neutrino) and matter (CDM + baryon) can be read from Eqs. (87) and (**??**) as

$$\dot{\delta}_r = -\frac{4}{3}\theta_r + 4\dot{\phi}\,, \quad \dot{\theta}_r = k^2\frac{1}{4}\delta_r + k^2\psi\,, \qquad (103)$$

$$\dot{\delta}_m = 3\phi' - \theta_m\,, \quad \dot{\theta}_m = -\mathcal{H}\theta_m + k^2\psi\,, \qquad (104)$$

where a subscript $_r$ denotes radiation while a subscript $_b$ denotes matter.

# Perturbations on large scales

- To solve the previous evolution equations, we use the perturbed Einstein equations to write $\psi$ and $\phi$ in terms of the density contrast and velocity perturbations.
- Adiabatic condition:

$$\delta_m = \frac{1}{4}\delta_r \,. \tag{105}$$

- We get

$$\psi = -\frac{1}{2}\delta_r = \text{constant}\,, \quad \rightarrow \quad \Delta_{T0} = -\frac{1}{2}\psi\,, \tag{106}$$

which describe the temperature perturbations due to the metric perturbations on the last scattering surface. This is an ordinary Sachs-Wolfe effect.

# Perturbations on large scales

- When the universe evolves through the matter-radiation equality the metric perturbation on large scales change.
- This gives the contribution to the ordinary Sachs-Wolfe effect, so that

$$[\Delta_{T0} + \psi] = \frac{1}{3}\psi, \tag{107}$$

which is the Sachs-Wolfe effect.

# free streaming

- After recombination, photons decouple from baryons and consequently freely propagate (free streaming) from the last scattering surface to observers at present.
- for $l \geq 2$, the eom for photon multipoles take the form of the recursion relation of the spherical Bessel function $j_l(x)$,

$$\frac{d}{dx} j_l(x) = \frac{l}{2l+1} j_{l-1}(x) - \frac{l+1}{2l+1} j_{l+1}(x)\,, \qquad (108)$$

$$\Rightarrow \quad \frac{d}{d\tau} j_l(k\tau) = \frac{l}{2l+1} k j_{l-1}(k\tau) - \frac{l+1}{2l+1} k j_{l+1}(k\tau)\,.$$

# free streaming

- By writing the eoms for monopole and dipole is suitable forms, we get

$$\Delta_{T\,l}(\tau,k) = \underbrace{[\Delta_{T0}+\psi]}_{\text{SW effect}}(\tau_s,k)\,j_l\,(k(\tau-\tau_s))$$
$$+\int_{\tau_s}^{\tau} d\tau'\,[\psi'(\tau')+\phi'(\tau')]\,j_l\,(k\tau-k\tau')\ , \qquad (109)$$

where a subscript $_s$ denotes evaluation on the Last scattering surface, the second term on the RHS of this equation is the contribution to the temperature perturbations from the time-dependence of the metric perturbations, and this contribution is the integrated Sachs-Wolfe (ISW) effect arising during the free streaming.

# free streaming

- The free streaming can also distribute oscillation of the temperature perturbations in photon-baryon fluid on the Last scattering surface to higher multipoles.
-
$$\Delta_{T\,l}\left(\tau_0, k\right) = \Delta_{T0\,,osc}\left(\tau_s, k\right) j_l\left(k\tau_*\right)\ , \tag{110}$$
where $\tau_* \equiv \tau_0 - \tau_s$.
- For the adiabatic case, $\Delta_{T0\,,osc}\left(\tau_s, k\right)$ has pakes at
$$k = \frac{n\pi}{r_s}\ , \quad n \in \{0, 1, 2, 3, ...\}\ . \tag{111}$$

# free streaming

- According to the maximum of the spherical Bessel function $j_l(k\tau_*)$ at $l \simeq k\tau_*$, the maximum of $\Delta^2_{T0osc}$ occurs around

$$l = n\frac{\pi\tau_*}{r_s} = nl_A \,, \qquad (112)$$

where $l_A$ is a characteristic acoustic index.

# The angular power spectra of CMB

- Photons, which free streming from the last scattering surface to observer at present, are redshifted due to the expansion of the universed such that the wavelengths of the photons are in the range of microwave.
- These photons are distributed almost smoothly in the universe, so that they can be viewed as the photons background in the universe known as the Cosmic Microwave Background (CMB).
- To connect predictions from the cosmological models with observations, the temperature perturbations of the CMB are quantified by power spectrum.

# The angular power spectra of CMB

- Since we observe the CMB photons at present and at our location in difference direction $\hat{n}$, we expand $\hat{n}$–dependence in terms of spherical harmonics $Y_{lm}(\hat{n})$ as

$$\Delta_T\left(\vec{x}_0, \hat{n}, \tau_0\right) = \sum_{l,m} a_{lm} Y_{lm}(\hat{n})\,, \tag{113}$$

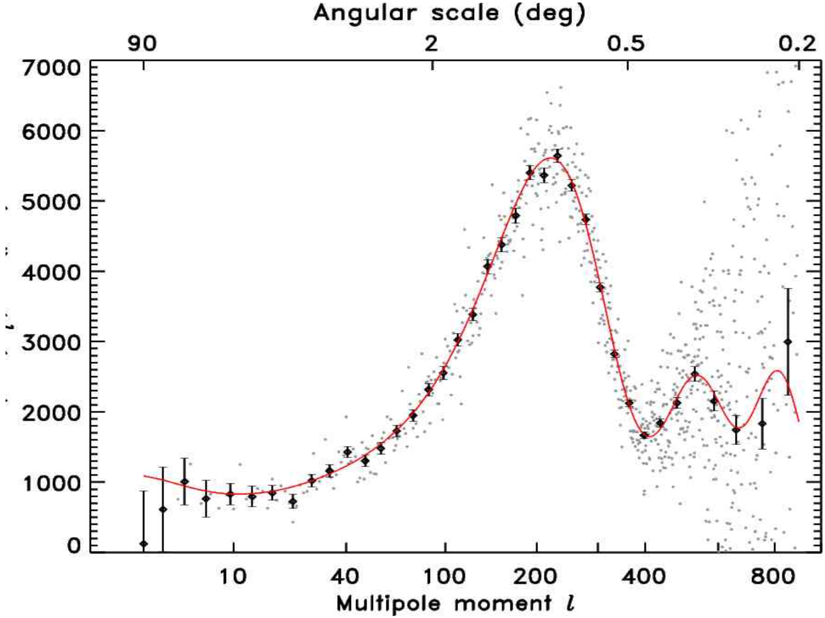
  where suvscript $_0$ denotes evaluation at present.

- We define the angular power spectrum $C_l$ of the CMB perturbation/anisotropy in terms of $a_{lm}$ as

$$\left\langle a_{lm} a^*_{l'm'} \right\rangle = \delta_{ll'} \delta_{mm'} C_l\,. \tag{114}$$

- Due to the statistical homogeneity and isotropy of the perturvation fields, the angular power spectrum $C_l$ does not depend on position and $m$, and $\left\langle a_{lm} a^*_{l'm'} \right\rangle$ vanishes for $l \neq l'$ or $m \neq m'$.

# The angular power spectra of CMB



# Line of Sight Integration

- The reference of this topic is [astro-ph/9603033]. We define differential optical depth for Thomson scattering as $\dot{\kappa} = a n_e \sigma_T$.
- When the polarization is included, the Boltzmann equation for photon perturbations becomes

$$\dot{\Delta}_T^{(S)} + ik\mu\Delta_T^{(S)} = \phi' - ik\mu\psi$$
$$+\dot{\kappa}\left[-\Delta_T^{(S)} + \Delta_{T0}^{(S)} + i\mu v_b + \frac{1}{2}P_2(\mu)\Pi\right], \quad (115)$$

where $\Delta_T^{(S)}$ is the scalar mode of the temperature perturbation.

- Here,

$$\Pi = \Delta_{T2}^{(S)} + \Delta_{P2}^{(S)} + \Delta_{P0}^{(S)}, \quad (116)$$

where $\Delta_P^{(S)}$ is the scalar mode of polarization in temperature perturbation.

# Line of Sight Integration

- The Boltzmann equation can be written in the integral form as

$$\Delta_T^{(S)} = \int_0^{\tau_0} d\tau \underbrace{e^{ik\mu(\tau-\tau_0)}}_{\text{angular dependent}} S_T^{(S)}(k,\tau)\,. \qquad (117)$$

- For the temperature perturbation, the source term is

$$\begin{aligned} S_T^{(S)}(k,\tau) &= g\left(\Delta_{T0} + \psi - \frac{v_b'}{k} - \frac{\Pi}{4} - \frac{3\Pi''}{4k^2}\right) \\ &+ \mathrm{e}^{-\kappa}\left(\phi' + \psi'\right) - g'\left(\frac{v_b}{k} + \frac{3\Pi'}{4k^2}\right) - \frac{3g''\Pi}{4k^2}\,. \end{aligned} \qquad (118)$$

# Line of Sight Integration

- We use the decomposition

$$e^{ik\mu(\tau-\tau_0)} = \sum_{l=0}^{\infty}(2l+1)(-i)^l j_l(k(\tau_0-\tau))P_l(\mu)\,. \quad (119)$$

- We finally get

$$\Delta^{(S)}_{(T,P)l}(k,\tau=\tau_0) = \int_0^{\tau_0} S^{(S)}_{T,P}(k,\tau) j_l[k(\tau_0-\tau)]d\tau, \quad (120)$$

- The advantage of (120) is the decomposition of $\Delta^{(S)}_{(T,P)l}$ in to $S^{(S)}_{T,P}$, which does not depend on the multipole moment $l$ and a geometrical term $j_l$, which does not depend on the particular cosmological model.