Essence of cosmological perturbation theory

Khamphee Karwan

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Perturbations in the universe

- According to cosmological principle, the universe is homogeneous and isotropic on large scales.
- In principle, the universe cannot be perfectly homogeneous and isotropic, because structures in the universe such as clusters of galaxies, galaxies, etc, cannot be created in the completely smooth universe.
- In the standard notion, structures in the universe are developed from small inhomogeneity and anisotropy in the early universe.
- The small inhomogeneity and anisotropy can be treated as perturbations around the homogeneous and isotropic universe.

Perturbation in the metric tensor

- In the subsequent topics, we will study how to quantify perturbations in spacetime and matter in the universe by decomposing metric and energy-momentum tensors into background and perturbed parts.
- The reference for this topic is [arXiv:astro-ph/0101563].

Perturbation in the metric tensor

- To describe small deviation from the homogeneity and isotropy of the spacetime, we decompose the metric tensor into the background part and perturbed part as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}\,, \tag{1}$$

  where $\bar{g}_{\mu\nu}$ and $\delta g_{\mu\nu}$ are the background and perturbed metrics.
- We will use an over bar to denote the homogeneous and isotropic background quantities.
- The metric $\bar{g}_{\mu\nu}$ is the FLRW metric, while $|\delta g_{\mu\nu}| < 1$.
- The components of the metric can be determined from the symmetry of the systems, e.g., $\bar{g}_{00} = -1, \bar{g}_{0i} = 0, \bar{g}_{ij} = a^2(t)\delta_{ij}$.

Perturbation in the metric tensor

- In the perturbed universe, there is no special symmetry to determine the components of the perturbed metric.
- Hence, we parameterize each component of the perturbed metric as follows:
- The component $0-0$ can be expressed in term of a scalar function $\psi(\tau,\vec{x})$ as

$$g_{00} = -a^2(\tau)\left(1 + 2\psi(\tau,\vec{x})\right) , \tag{2}$$

where $\tau = \int dt/a$ is the conformal time and $|\psi| < 1$ denotes deviation from FLRW metric.

Perturbation in the metric tensor

- The components $0 - i$ and $i - 0$ can be expressed in term of a three-dimensional vector.

$$g_{i0} = g_{0i} = a^2(\tau) v_i(\tau, \vec{x}) \,. \tag{3}$$

Note that there is no background part in g_{0i}.

- From the vector analysis, any vector field can be decomposed into curl- and divergence-free parts, so that

$$g_{i0} = g_{0i} = a^2(\tau) \left(B_{,i}(\tau, \vec{x}) - S_j(\tau, \vec{x}) \right) \,, \tag{4}$$

where subscript $_{,i}$ denotes $\partial/\partial x^i$ and $S_{i,i} = 0$.

- We see that g_{i0} can be expressed in terms of a scalar function and divergence-free vector.

Perturbation in the metric tensor

- The component g_{ij} can be expressed in terms of scalar function as

$$g_{ij} = a^2(\tau)\left[(1 - 2\phi(\tau,\vec{x}))\,\delta_{ij} + 2E_{,ij}(\tau,\vec{x})\right] . \qquad (5)$$

- This component of the metric can also be expressed in terms of the divergence-free vector F_i as

$$g_{ij} = a^2(\tau)\left(F_{i,j}(\tau,\vec{x}) + F_{j,i}(\tau,\vec{x})\right) . \qquad (6)$$

- The last part of h_{ij} is the three-dimensional tensor which is tressless and divergence-free :

$$d_{ij} = a^2(\tau) H_{ij}(\tau,\vec{x}) , \quad \text{where} \quad H^i_i = H^{ij}_{,j} = 0 . \qquad (7)$$

Perturbation in the metric tensor

- Hence, we have

$$g_{ij} = a^2 \left[(1 - 2\phi)\delta_{ij} + 2E_{,i,j} + F_{i,j} + F_{j,i} + H_{ij} \right] . \qquad (8)$$

- The perturbed metric can be parameterized by four scalar functions, ψ, ϕ, B, E, two divergence-free vectors, S_i, F_i and a traceless and divergence-free tensor H_{ij}.
- There are four degrees of freedom from four scalar functions, four degrees of freedom from two divergence-free vectors, and two degrees of freedom from tressless and divergence-free tensor, so that we have ten degrees of freedom in total.
- These scalar, 3-D vector and 3-D tensor fields completely characterized components of the metric tensor.

Perturbation in the metric tensor

- For linear perturbations, these scalar, vector, and tensor fields evolve independently.
- This means that the perturbation in metric tensor can be decomposed into scalar, vector, and tensor perturbations (or modes).

Perturbation in the energy-momentum tensor

- The energy-momentum tensor of the perfect fluid can be written in the general form as

$$T_{\mu\nu} = (\rho + P) u_\mu u_\nu + g_{\mu\nu} P \,, \tag{9}$$

where ρ, P, and u^μ are the energy density, pressure, and four-velocity of the fluid.

- The energy density and pressure are scalar quantities, they can be decomposed into a background and a perturbed parts as

$$\rho = \bar{\rho}(\tau) + \delta\rho(\tau, \vec{x}) \,, \quad \text{and} \quad P = \bar{P}(\tau) + \delta P(\tau, \vec{x}) \,. \tag{10}$$

- In the homogeneous and isotropic background, the spatial component of the four-velocity must vanish, i.e.,

$$u^\mu = \frac{1}{a} (1, 0, 0, 0) \,, \quad u_\mu = a \left(-1, 0, 0, 0 \right) \,, \tag{11}$$

Perturbation in the energy-momentum tensor

- The temporal and spatial components of the four-velocity can also be expressed in terms of scalar functions and divergence-free vector V^i, so that

$$u^\mu = \frac{1}{a}\left(u^0, v^{,i} + V^i\right) \,, \quad u_\mu = a\left(u_0, v_{,i} + V_i\right) \,, \tag{12}$$

- The temporal component of u^μ is estimated from

$$g_{\mu\nu} u^\mu u^\nu = -1 \,, \quad \to \quad u^0 = \frac{1}{a}(1-\psi) + \text{higher order} \,. \tag{13}$$

- Hence, we have

$$u^\mu = \frac{1}{a}\left(1-\psi, v^{,i} + V^i\right) \,, \quad u_\mu = a\left(-1-\psi, v_{,i} + V_i\right) \,, \tag{14}$$

Perturbation in the energy-momentum tensor

- The energy-momentum tensor can be expressed up to the first order perturbations as

$$T_0^0 = -(\bar{\rho} + \delta\rho) , \quad (15)$$

$$T_i^0 = (\bar{\rho} + \bar{P})(B_{,i} + v_{,i} + V_i - S_i) , \quad (16)$$

$$T_0^i = -(\bar{\rho} + \bar{P})(v^{,i} + V^i) , \quad (17)$$

$$T_j^i = (\bar{P} + \delta P)\,\delta_j^i . \quad (18)$$

Perturbation in the energy-momentum tensor

- For general fluid, the component i-j of the energy-momentum tensor can contain traceless and divergence-free parts:

$$T^i_j = \left(\bar{P} + \delta P\right) \delta^i_j + \pi^i_j \, , \tag{19}$$

  where π^i_j describes anisotropy in the spatial part of the energy-momentum tensor.

- The anisotropic perturbation π^i_j can be decomposed into scalar, vector, and tensor parts as

$$\pi^i_j = \underbrace{\Pi^{,i}_{,j} - \frac{1}{3}\Delta^2 \Pi \delta^i_j}_{\text{traceless}} + \frac{1}{2}\left(\pi^i_{,j} + \pi_j^{,i}\right) + \Pi^i_j \, , \tag{20}$$

  where $\Delta^2 \equiv \partial_i \partial^i$, $\pi^i_{,i} = \Pi^i_i = \Pi^i_{j,i} = 0$.

Gauge degrees of freedom

- Inserting the perturbed metric and perturbed energy-momentum tensors into the Einstein equation, we get two evolution equations and two constraint equations as follows:
- Components $0-0$ and $0-i$ yield energy and momentum constraint equations

$$3\mathcal{H}\left(\dot{\phi}+\mathcal{H}\psi\right)-\Delta^2\phi-\mathcal{H}\Delta^2\sigma=-\frac{m_p^2}{2}a^2\delta\rho\,, \quad (21)$$

$$\dot{\phi}+\mathcal{H}\psi=-\frac{m_p^2}{2}a^2\left(\bar{\rho}+\bar{P}\right)(v+B)\,, \quad (22)$$

where a dot denotes the derivative with respect to the conformal time, $\mathcal{H}\equiv\dot{a}/a$, and $\sigma\equiv-B+\dot{E}$ is the shear perturbation.

Gauge degrees of freedom

- Components $i-i$ and $i \neq j$ yield two evolution equations

$$\ddot{\phi}+2\mathcal{H}\dot{\phi}+\mathcal{H}\dot{\psi}+\left(2\dot{\mathcal{H}}+\mathcal{H}^2\right)\psi=\frac{m_p^2}{2}a^2\delta P\,, \quad (23)$$

$$\dot{\sigma}_s+2\mathcal{H}\sigma_s-\psi+\phi=m_p^2a^2\Pi\,, \quad (24)$$

- The temporal and spatial components of $\nabla_\nu T^\nu_\mu=0$ give the conservation equations for energy and momentum

$$\delta\dot{\rho}+3\mathcal{H}(\delta\rho+\delta P)=(\bar{\rho}+\bar{P})\left[3\dot{\phi}-\Delta^2(v+\dot{E})\right], \quad (25)$$

$$\frac{\partial}{\partial\tau}\left[(\bar{\rho}+\bar{P})(v+B)\right]+\delta p+\frac{2}{3}\Delta^2\Pi = -(\bar{\rho}+\bar{P})\left[\psi+4\mathcal{H}(v+B)\right]\,. \quad (26)$$

Gauge degrees of freedom

- For scalar perturbation, the metric perturbation can be described by four scalar functions, ϕ, B, E and ψ.
- The perturbation in the energy-momentum tensor can also be quantified by four scalar functions, $\delta\rho, \delta P, v$ and Π.
- However, if we combine the evolution equations from $\nabla\nu T^{\nu}_{\mu} = 0$, and the perturbed Einstein equations, only five equations are linearly independent.
- In general, the relation between $\delta\rho$ and δP is given by

$$\delta P = c_s^2 \delta\rho \,, \tag{27}$$

where c_s^2 is the sound speed square of the perturbations.

Gauge degrees of freedom

- This suggests that two degrees of freedom are not physical degrees of freedom. They are gauge degrees of freedom.
- The gauge degrees of freedom are consequences of diffeomorphism invariance of GR.
- The gauge degrees of freedom can be eliminated by performing calculations in suitable hypersurfaces or by using gauge-invariant variables which are suitable combinations of perturbed variables.
- In this lecture, we briefly discuss how to eliminate gauge degrees of freedom by choosing suitable hypersurfaces, and we will focus on scalar perturbations.

Gauge degrees of freedom

- The gauge transformations in GR are the coordinate transformations.
- The infinitesimal coordinate transformations can be written as

$$\tilde{\tau} = \tau + \xi^0(\tau, x^i)\,, \quad \tilde{x}^i = x^i + \xi^{,i}(\tau, x^i)\,, \qquad (28)$$

where ξ^0 and ξ are gauge degrees of freedom. These quantities are the perturbed quantities because they depend on spatial coordinates.

Gauge degrees of freedom

- The metric tensor can be written in terms of the line element ds^2 which is a scalar quantity as

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= a^2(\tau)\{ -(1+2\psi)d\tau^2 + 2B_{,i} d\tau dx^i \\ &\quad +[(1-2\phi)\delta_{ij} + 2E_{,ij}]dx^i dx^j\} . \end{aligned} \tag{29}$$

- We now consider gauge transformation of this line element.
- Up to the first order in the coordinate transformations, the infinitesimal coordinate transformations of the gauge functions are

$$\xi^0(\tau, x^i) = \xi^0(\tilde{\tau}, \tilde{x}^i) , \quad \xi(\tau, x^i) = \xi(\tilde{\tau}, \tilde{x}^i) , \tag{30}$$

Gauge degrees of freedom

- The relation between the gauge function at different coordinates can be computed as

$$\begin{aligned}
\xi^0(\tau, x^i) &= \xi^0(\tilde{\tau}, \tilde{x}^i) + \left.\frac{\partial \xi^0}{\partial \tau}\right|_* (\tau - \tilde{\tau}) + \left.\frac{\partial \xi^0}{\partial x^j}\right|_* (x^j - \tilde{x}^j) + \ldots \\
&= \xi^0(\tilde{\tau}, \tilde{x}^i) - \left.\frac{\partial \xi^0}{\partial \tau}\right|_* \xi^0(\tau, x^i) - \left.\frac{\partial \xi^0}{\partial x^j}\right|_* \left.\frac{\partial \xi}{\partial x_j}\right|_0 + \ldots \\
&= \xi^0(\tilde{\tau}, \tilde{x}^i) + \text{higher order}\,. \qquad (31)
\end{aligned}$$

where subscripts $_*$ and $_0$ denote evaluation at $(\tilde{\tau}, \tilde{x}^i)$ and (τ, x^i).

Gauge degrees of freedom

- From the gauge transformation:

$$\tilde{\tau} = \tau + \xi^0(\tau, x^i)\,, \quad \tilde{x}^i = x^i + \xi^{,i}(\tau, x^i)\,. \tag{32}$$

- The inverse coordinate transformations are

$$\tau = \tilde{\tau} - \xi^0(\tau, x^i) = \tilde{\tau} - \xi^0(\tilde{\tau}, \tilde{x}^i)\,, \quad x^i = \tilde{x}^i - \xi^{,i}(\tilde{\tau}, \tilde{x}^i)\,, \tag{33}$$

- Hence, we have

$$d\tau = d\tilde{\tau} - \dot{\xi}^0 d\tilde{\tau} - \xi^0_{,i} d\tilde{x}^i\,, \quad dx^i = d\tilde{x}^i - \dot{\xi}^{,i} d\tilde{\tau} - \xi^{,i}_{,j} d\tilde{x}^j\,. \tag{34}$$

Gauge degrees of freedom

- The calculation is

$$d\tau = d\tilde{\tau} - \left.\frac{\partial\xi^0}{\partial\tilde{\tau}}\right|_* d\tilde{\tau} - \left.\frac{\partial\xi^0}{\partial\tilde{(x)}^i}\right|_* d\tilde{x}^i$$
$$= d\tilde{\tau} - \dot{\xi}^0 d\tilde{\tau} - \xi^0_{,i} d\tilde{x}^i \,. \qquad (35)$$

- For the scale factor, we have

$$a(\tau) = a(\tilde{\tau}) + \dot{a}|_{\tilde{\tau}}(tau - \tilde{\tau}) = a(\tilde{\tau}) - \xi^0 \dot{a}(\tilde{\tau}) \,. \qquad (36)$$

Gauge degrees of freedom

- Each part of The line element in the new coordinates is transformed as

$$
\begin{aligned}
&-a^2(\tau)(1+2\psi)d\tau^2 \\
&= -a^2(1-2\xi^0\mathcal{H})(1+2\psi)(d\tilde{\tau}^2 - 2\dot{\xi}^0 d\tilde{\tau}^2 - 2\xi^0_{,i} d\tilde{x}^i d\tilde{\tau})\,.
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
&2a^2(\tau)B_{,i}d\tau dx^i \\
&= 2a^2 B_{,i} d\tilde{\tau} d\tilde{x}^i + \text{higher order}\,.
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
&a^2(1-2\xi^0\mathcal{H})\left[(1-2\phi)\delta_{ij} + 2E_{,ij}\right] dx^i dx^j \\
&= a^2(1-2\xi^0\mathcal{H})\left[(1-2\phi)\delta_{ij} + 2E_{,ij}\right] \times \\
&(d\tilde{x}^i d\tilde{x}^j - 2\dot{\xi}^{,i} d\tilde{\tau} d\tilde{x}^j - 2\xi^{,i}_{,k} d\tilde{x}^k d\tilde{x}^j)\,.
\end{aligned} \tag{39}
$$

Gauge degrees of freedom

- Combining all parts, we get

$$
\begin{aligned}
ds^2 &= -a^2\left[1+2(\underbrace{\psi-\xi^0\mathcal{H}-\dot{\xi}^0}_{=\tilde{\psi}})\right]d\tilde{\tau}^2+2a^2(\underbrace{B_{,i}+\xi^0_{,i}-\dot{\xi}_{,i}}_{=\tilde{B}_{,i}}) \\
&+ a^2\left[\left(1-2(\underbrace{\phi+\xi^0\mathcal{H}}_{=\tilde{\phi}})\delta_{ij}\right)+2(\underbrace{E_{,ij}-\xi_{,ij}}_{=\tilde{E}_{,ij}})\right]d\tilde{x}^i d\tilde{x}^j
\end{aligned}
$$

Gauge degrees of freedom

- Under gauge transformations, the perturbation variables are transformed as

$$\tilde{\psi} = \psi - \mathcal{H}\xi^0 - \dot{\xi}^0\,, \quad \tilde{B} = B + \xi^0 - \dot{\xi}\,, \tag{41}$$
$$\tilde{\phi} = \phi + \mathcal{H}\xi^0\,, \quad \tilde{E} = E - \xi\,. \tag{42}$$

- If we perform the calculation on hypersurfaces where $\tilde{E} = 0$, the spatial gauge degree of freedom can be fixed such that $\xi = E$.
- Under the gauge transformation,

$$\tilde{B} - \dot{\tilde{E}} = B - \dot{E} + \xi^0\,. \tag{43}$$

Gauge degrees of freedom

- The temporal gauge degree of freedom can be fixed such that $\xi^0 = -B + \dot{E}$ if we perform calculation onhypersurfaces where $\tilde{B} - \dot{\tilde{E}} = 0$.
- This means that if we work in the hypersurface on which $\tilde{B} = \tilde{E} = 0$, both temporal and spatial gauge degrees of freedom are completely fixed.
- The hypersurface on which $\tilde{B} = \tilde{E} = 0$ can be reached from any hypersurfaces if we choose $\xi = E$ and $\xi^0 = \dot{E} - B$.
- This gauge choice is the Conformal Newtonian Gauge.
- We see that two gauge degrees of freedom could be eliminated/fixed if we set two of the perturbation variables to zero.

Gauge degrees of freedom

- Fixing the gauge can also be done by choosing the gauge degrees of freedom based on the gauge transformation properties of the perturbations in a fluid.
- In general, the gauge degrees of freedom may not be completely fixed even though two perturbation variables are set to zero.
- One of the gauge choices, which is often used in numerical integration, is the Synchronous gauge in which the perturbations in the temporal components of the metric vanish, i.e., $\tilde{\psi} = \tilde{B} = 0$.
- For the Synchronous gauge, we have

$$\xi^0 = \frac{1}{a} \int d\tau a \psi + \frac{f(x^i)}{a} \,, \tag{44}$$

where $f(x^i)$ is an arbitrary function of spatial coordinates which is a residual gauge freedom.

Gauge degrees of freedom

- To fix a residual gauge freedom, we have to put additional conditions by performing calculations in a special frame of fluid, e.g., a rest frame of cold dark matter.
- In the subsequent topics, we will present the evolution equations for the linear perturbations in the universe.
- The reference for these topics is [arXiv:astro-ph/9506072].

Evolution equations for perturbations

- In the following studies, we will focus on the Newtonian gauge in which the line element takes the form

$$ds^2 = a^2(\tau)\left\{-(1+2\psi)d\tau^2 + (1-2\phi)dx^i dx_i\right\} . \quad (45)$$

- The following equations will be expressed in the Fourier space

$$\psi(\tau,\vec{x}) = \int d^3k e^{i\vec{k}\cdot\vec{x}}\psi(\tau,\vec{k}) \quad (46)$$

- According to our reference, velocity perturbation is described in terms of its divergence, and the anisotropic perturbation is described in terms of shear stress:

$$(\bar{\rho}+\bar{P})\theta \equiv ik^j \delta T^0{}_j\,, \quad (\bar{\rho}+\bar{P})\sigma \equiv -(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij})\Sigma^i{}_j\,, \quad (47)$$

where $\hat{k} = \vec{k}/|\vec{k}|$.

Evolution equations for perturbations

- Here, the traceless component of $T^i{}_j$ is

$$\Sigma^i{}_j \equiv T^i{}_j - \frac{1}{3}\delta^i{}_j T^k{}_k \,, \tag{48}$$

- The perturbed Einstein equations in the Newtonian gauge yield

$$3\mathcal{H}\left(\dot{\phi} + \mathcal{H}\psi\right) + k^2\phi = -\frac{m_p^2}{2}a^2\delta\delta\rho \,, \tag{49}$$

$$k^2\left(\dot{\phi} + \mathcal{H}\psi\right) = \frac{m_p^2}{2}a^2(\bar{\rho} + \bar{P})\theta \,, \tag{50}$$

$$\ddot{\phi} + \mathcal{H}(\dot{\psi} + 2\dot{\phi}) + \left(2\dot{\mathcal{H}} + \mathcal{H}^2\right)\psi$$

$$+\frac{k^2}{3}(\phi - \psi) = \frac{m_p^2}{2}a^2\delta P \,, \tag{51}$$

$$k^2(\phi - \psi) = \frac{3m_p^2}{2}a^2(\bar{\rho} + \bar{P})\sigma \,, \tag{52}$$

Evolution equations for CDM perturbations

- The perturbation in energy density $\delta\rho$ is described by the dimensionless density contrast $\delta \equiv \delta\rho/\bar{\rho}$.
- The cold dark matter (CDM) can be described by a perfect fluid, so that the evolution equations for its perturbations can be computed from $\nabla_\nu T^\nu_\mu = 0$:

$$\dot{\delta}_c = -\theta_c + 3\dot{\phi}\,, \quad \dot{\theta}_c = -\frac{\dot{a}}{a}\theta_c + k^2\psi\,, \qquad (53)$$

  where subscript $_c$ denotes CDN.
- During some epochs, the mean free path of photons and neutrinos is very long, so that the perturbations cannot be described by perfect fluid.

Evolution equations for perturbations in radiation

- To describe perturbations in photon and neutrino, we use the distribution function which is defined on the phase space.
- The phase space is spanned by conjugate momentum and coordinates.
- The conjugate momentum is the spatial component of the 4-momentum with lower indices denoted by P_i.
- To remove the contribution from the metric tensor, it is convenient to work with momentum in the orthogonal bases p_i defined as

$$\eta^{\mu\nu} p_\mu p_\nu = -m^2 = g^{\mu\nu} P_\mu P_\nu \,. \tag{54}$$

Here, $\eta^{\mu\nu}$ is the Minkowski metric, and m is a rest mass of a particle.

Evolution equations for perturbations in radiation

- The relation between P_μ and p_μ is given by

$$P_\mu = \mathrm{e}_\mu^\alpha p_\alpha \,, \quad \text{so that} \quad g^{\mu\nu}\mathrm{e}_\mu^\alpha \mathrm{e}_\nu^\beta = \eta^{\alpha\beta} \,, \qquad (55)$$

  where e_μ^α is a tetrad.

- For the Newtonian gauge, we get

$$P_0 = (1+\psi)p_0 \,, \quad P_i = \mathrm{e}_i^j p_j = a(1-\phi)p_i \,. \qquad (56)$$

- The energy of a particle defined in orthogonal bases is

$$\epsilon = p_0 = \sqrt{p_i p^i + m^2} = \sqrt{p^2 + m^2} \,, \quad \rightarrow \quad P_0 = -(1+\psi)\epsilon \,. \qquad (57)$$

Evolution equations for perturbations in radiation

- An infinitesimal volume of phase space is $dV = dx^1 dx^2 dx^3 dP_1 dP_2 dP_3$, and the number of particles in a unit volume is

$$dN = f(x^i, P_j, P_0, \tau) dV \,. \tag{58}$$

- In the background universe, the distribution function is the Fermi-Dirac distribution for fermions and the Bose-Einstein distribution for bosons given by

$$f_0(\epsilon) = \frac{g_s}{2\pi^2} \frac{1}{\mathrm{e}^{\epsilon/T_0} \pm 1} \,, \tag{59}$$

where $T_0 = aT$ denotes the present temperature of the particles, the factor g_s is the number of spin states.

Evolution equations for perturbations in radiation

- The energy-momentum tensor can be written in terms of the distribution function and the 4-momentum components as

$$T_{\mu\nu} = \int dP_1 dP_2 dP_3 \, (-g)^{-1/2} \, \frac{P_\mu P_\nu}{P^0} f(x^i, P_j, \tau) \, , \quad (60)$$

- In the perturbed universe, it is convenient to quantify the momentum in terms of

$$q_j \equiv a p_j \, , \quad (61)$$

- Moreover, we define the direction of the momentum q_i through the unit vector n_i by

$$q_j = q n_j \, , \quad \text{where} \quad n^i n_j = \delta^i_j \, . \quad (62)$$

Evolution equations for perturbations in radiation

- The distribution function can be decomposed into the background and perturbed parts as

$$f(x^i, P_j, P_0, \tau) = f_0(q)\left[1 + \Psi(x^i, q, n_j, \tau)\right] , \qquad (63)$$

where we have used $\epsilon = q$ for massless particles.

- The perturbations in $(-g)^{-1/2}$ and the volume element are

$$(-g)^{-1/2} = a^{-4}(1-\psi+3\phi) , \quad dP_1 dP_2 dP_3 = (1-3\phi) q^2 dq d\Omega , \qquad (64)$$

where $d\Omega$ is the solid angle associated with direction n_i.

- For example, the component 0-0 of the energy-momentum tensor is

$$T^0{}_0 = \int dP_1 dP_2 dP_3 \, (-g)^{-1/2} \frac{P_0 P^0}{P^0} f = -a^{-4} \int q^3 dq \, d\Omega \, f_0(q) \qquad (65)$$

Evolution equations for perturbations in radiation

- For the background, we have

$$\bar{\rho} = a^{-4} 4\pi \int q^3 dq \, f_0(q) \, . \tag{66}$$

- For the perturbation, we get

$$\delta\rho = 3\delta P = a^{-4} \int q^2 dq d\Omega \, q f_0(q) \Psi \, . \tag{67}$$

- In the Fourier space, the distribution function becomes

$$f(x^i, q, n_j, \tau) \rightarrow f(k^i, q, n_j, \tau) \, , \tag{68}$$

where k^i is a wavenumber of the Fourier modes.

Evolution equations for perturbations in radiation

- We can integrate out q from the distribution function, and expand the angular-dependent part of the resulting function in a series of Legendre polynomials $P_l(\hat{k}\cdot\hat{n})$ as

$$F(\vec{k},\hat{n},\tau) \equiv \frac{\int q^3 dq f_0(q)\Psi}{\int q^3 dq f_0(q)} = \sum_{l=0}^{\infty}(-i)^l(2l+1)F_l(\vec{k},\tau)P_l(\mu)\,, \tag{69}$$

where $\mu \equiv \hat{k}\cdot\hat{n} = \cos\theta$.

- For example:

$$\begin{aligned}\delta &\equiv \frac{\delta\rho}{\bar{\rho}} = \int d\Omega \frac{\int q^3 dq f_0(q)\Psi}{4\pi \int q^3 dq\, f_0(q)} \\ &= \frac{1}{4\pi}\int d\Omega F(\vec{k},\hat{n},\tau) \\ &= \frac{1}{2}\int d\mu \sum_{l=0}^{\infty}(-i)^l(2l+1)F_l(\vec{k},\tau)P_l(\mu) = F_0\,,\end{aligned}$$

Evolution equations for perturbations in radiation

- In the previous calculation, we have used $d\Omega = 2\pi \sin\theta d\theta = -2\pi d\cos\theta = -2\pi d\mu$, and

$$\int_{-1}^{1} P_l(x) P_m(x)\, dx = \frac{2}{2l+1}\delta_{lm}\,. \tag{70}$$

- For θ and σ, one can shows that

$$\theta = \frac{3}{4} k F_1\,, \quad \sigma = \frac{1}{2} F_2\,. \tag{71}$$

Evolution equations for perturbations in radiation

- The evolution of the distribution function is described by the Boltzmann equation which is given in real space by

$$\frac{\partial f}{\partial \tau} + \frac{dx^i}{d\tau}\frac{\partial f}{\partial x^i} + \frac{dq}{d\tau}\frac{\partial f}{\partial q} + \frac{dn_i}{d\tau}\frac{\partial f}{\partial n_i} = \left(\frac{\partial f}{\partial \tau}\right)_C , \qquad (72)$$

  where the term on the RHS corresponds to the collision between the group of particles described by the distribution function and the other particles.

- The term $dx^i/d\tau$ can be written as

$$\frac{dx^i}{d\tau} = \frac{d\lambda}{d\tau}\frac{dx^i}{d\lambda} = \frac{d\lambda}{d\tau}P^i , \qquad (73)$$

  where λ is an affine parameter.

- We can compute $d\lambda/d\tau$ as

$$P^\alpha = \frac{dx^\alpha}{d\lambda} , \quad \rightarrow \quad \frac{d\lambda}{d\tau} = \frac{1}{P^0} . \qquad (74)$$

Evolution equations for perturbations in radiation

- Hence,

$$\frac{dx^i}{d\tau} = \frac{d\lambda}{d\tau}\frac{dx^i}{d\lambda} = \frac{P^i}{P^0} = \frac{qn^i}{\epsilon}\,, \tag{75}$$

  where we keep the first order in perturbation.

- The term $dq/d\tau$ can be computed from the geodesic equation

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0\,. \tag{76}$$

- Using $d\lambda/d\tau$ on the previous page, the above equation becomes

$$P^0\frac{dP^\mu}{d\tau} + \Gamma^\mu{}_{\alpha\beta}P^\alpha P^\beta = 0\,, \tag{77}$$

Evolution equations for perturbations in radiation

- Hence, we get

$$\frac{dq}{d\tau} = q\dot{\phi} - \epsilon(q,\tau)\, n_i \partial_i \psi \,. \tag{78}$$

- The terms $dn_i/d\tau$ and $\partial f/\partial n_i$ are first order in perturbation, so that we ignore the multiplication of these terms.
- The Boltzmann equation in the Fourier space is

$$\frac{\partial \Psi}{\partial \tau} + i\,\frac{q}{\epsilon}\,(\vec{k}\cdot\hat{n})\,\Psi + \frac{d\ln f_0}{d\ln q}\left[\dot{\phi} - i\,\frac{\epsilon}{q}(\vec{k}\cdot\hat{n})\,\psi\right] = \frac{1}{f_0}\,\left(\frac{\partial f}{\partial \tau}\right)_C \,. \tag{79}$$

Evolution equations for perturbations in neutrino

- Let us first consider the case of massless neutrinos, for this case, there is no collision term, so that

$$\frac{\partial \Psi}{\partial \tau} + i\,\frac{q}{\epsilon}\,(\vec{k}\cdot\hat{n})\,\Psi + \frac{d\ln f_0}{d\ln q}\left[\dot{\phi} - i\,\frac{\epsilon}{q}(\vec{k}\cdot\hat{n})\,\psi\right] = 0\,. \quad (80)$$

  In the following calculation, we will set $\epsilon = q$.

- We integrate the above equation over $q^3 dq$ and divide the result by $\int q^3 dq$ as

$$\int q^3 dq \left\{\frac{\partial f_0 \Psi}{\partial \tau} + ik\mu f_0 \Psi + q\frac{df_0}{dq}\left[\phi' - ik\mu\psi\right]\right\} = 0\,,$$

$$\frac{\partial F}{\partial \tau} + ikP_1(\mu)F + \underbrace{\int q^4 dq \frac{df_0}{dq}}_{=-4\int q^3 dq f_0} \frac{\left[\phi' - ikP_1(\mu)\psi\right]}{\int q^3 f_0 dq} = 0\,,$$

$$\frac{\partial F}{\partial \tau} + ikP_1(\mu)F - 4P_0(\mu)\left[\phi' - ikP_1(\mu)\psi\right] = 0\,. \quad (81)$$

- To extract the evolution equations for the multipole

Evolution equations for perturbations in neutrino

- The coupled differential equations for the multipoles expansion of neutrino distribution:

$$
\begin{aligned}
\dot{\delta}_\nu &= -\frac{4}{3}\theta_\nu + 4\dot{\phi}\,, \\
\dot{\theta}_\nu &= k^2\left(\frac{1}{4}\delta_\nu - \sigma_\nu\right) + k^2\psi\,, \\
\dot{F}_{\nu\, l} &= \frac{k}{2l+1}\left[lF_{\nu\,(l-1)} - (l+1)F_{\nu\,(l+1)}\right]\,, \quad l \geq 2\,.
\end{aligned}
\tag{82}
$$

Evolution equations for perturbations in photon

- We now consider the Boltzmann equation for the photon.
- For the photon, there is a coupling between photons and charged particles.
- The coupling arise from the Compton scattering between photons and electrons and the effects of scattering are transferred to baryons through the Coulomb interaction between baryons and electrons.
- Since the mass of the charged baryons (protons) is much larger than the mass of electrons, the main contribution to the scattering process comes from baryons (through baryon velocity).
- Hence, this scattering process is usually called the coupling between photons and baryons.

Evolution equations for perturbations in photon

- the Boltzmann equation for photons is

$$\frac{\partial F}{\partial \tau} + ik(\mu F - 4\left[\phi' - ik\mu\psi\right]$$
$$= a\sigma_T n_e \left[F_0 + 4ik\mu v_b - F - \frac{1}{2}F_2 P_2\left(\mu\right)\right] . \quad (83)$$

Evolution equations for perturbations in photon

- The EOM for the multipole moments of photon perturbations are

$$
\begin{aligned}
\dot{\delta}_\gamma &= -\frac{4}{3}\theta_\gamma + 4\dot{\phi}\,, \\
\dot{\theta}_\gamma &= k^2\left(\frac{1}{4}\delta_\gamma - \sigma_\gamma\right) + k^2\psi - an_e\sigma_T(\theta_\gamma - \theta_b)\,, \\
\dot{F}_{\gamma\,2} &= 2\dot{\sigma}_\gamma = \frac{8}{15}\theta_\gamma - \frac{3}{5}kF_{\gamma\,3} \\
&\quad -\frac{9}{5}an_e\sigma_T\sigma_\gamma + \frac{1}{10}an_e\sigma_T\left(G_{\gamma\,0} + G_{\gamma\,2}\right)\,, \\
\dot{F}_{\gamma\,l} &= \frac{k}{2l+1}\left[lF_{\gamma\,(l-1)} - (l+1)F_{\gamma\,(l+1)}\right] \\
&\quad -an_e\sigma_T F_{\gamma\,l}\,, \quad l \geq 3
\end{aligned}
\tag{84}
$$

Evolution equations for perturbations in baryon

- For baryon, we have to add the contribution from energy-momentum transfer due to photon coupling to the evolution equation for θ_b, such that the energy and momentum of photon-baryon fluid are conserved.
- The evolution equations for perturbations in baryon are

$$\dot{\delta}_b = 3\dot{\phi} - \theta_b \,, \tag{85}$$

$$\dot{\theta}_b = -\mathcal{H}\theta_b + k^2 c_b^2 \delta_b + k^2 \psi + \frac{4\bar{\rho}_\gamma}{3\bar{\rho}_b} a n_e \sigma_T (\theta_\gamma - \theta_b) \,, \tag{86}$$

where c_b^2 is a baryon sound speed.

The important processes for CMB

- In the early times (the age of the universe ¡ 300,000 years), the mean energy of photons is large.
- As a result, free electrons and nuclei cannot be combined to form stable neutral atoms, consequently, the number density of free electrons is large.
- During this epoch, photons and baryons are tightly coupled.
- The coupling arises from the Compton scattering between photons and electrons and the effects of scattering are transferred to baryons through the Coulomb interaction between baryons and electrons.

The important processes for CMB

- Due to the tight coupling, the mean free path between photons and baryons is very short such that they move together which can be described by a single fluid called photon-baryon fluid, and the higher multipole moment of photon perturbation is suppressed.
- However, on length scales that are larger than the mean free path between photons and baryons, the photons can diffuse through the baryons
- As a result, the perturbations in energy density of the photon-baryon fluid evolve like the force oscilation.
- When the universe further expands, the mean energy of photons reduces, consequently, stable neutral atoms can be formed.
- At the recombination epoch, the number density of free electrons reduces rapidly due to the formation of the neutral atoms, as a result, photons can freely propagate.

The important processes for CMB

- The epoch in which most of the photons last scatter with the charged particles before freely propagating to the observer at present is the last scattering surface.
- During the free propagation of the photons, the perturbation in the energy density of the photon-baryon fluid generates higher multipole moments of photon perturbations due to the free-streaming process.
- The free-streaming leads to an oscillation pattern in CMB angular power spectrum.
- The oscillation of the perturbations in energy density of the photon-baryon fluid at the last scattering surface leads to the oscillation pattern in the power spectrum of matter.

The important processes for CMB

- The freely propagating photons are redshifted due to the expansion of the universe such that the wavelengths of the photons are in the range of microwave and infrared when they reach the observer at present.
- These photons are distributed almost smoothly in the universe, so that they can be viewed as the photons background in the universe known as the Cosmic Microwave Background (CMB).
- To connect predictions from the cosmological models with observations, the temperature perturbations of the CMB are quantified by the power spectrum.

The angular power spectrum of CMB

- To study perturbations in photons, we expand the angular part of the distribution function as

$$F(\vec{k}, \hat{n}, \tau) = \sum_{l=0}^{\infty} (-i)^l (2l+1) F_l(\vec{k}, \tau) P_l(\mu) \,, \qquad (87)$$

- Since $\delta = F_0$ and $\delta = 4\delta T/\bar{T} = \Delta_T$, we define $F(\vec{k}, \hat{n}, \tau) \equiv 4\Delta_T(\vec{k}, \hat{n}, \tau)$.
- The relation $\delta = 4\delta T/\bar{T} = \Delta_T$ is computed from $\rho \propto T^4$.

The angular power spectrum of CMB

- Hence, the angular-dependent of the temperature perturbation in photon can be expanded as

$$\Delta_T(\vec{k}, \hat{n}, \tau) = \sum_{l=0}^{\infty} (-i)^l (2l+1) \Delta_{T\,l}(\vec{k}, \tau) P_l(\mu) \,, \quad (88)$$

- In the observation of CMB, we observe the CMB photons at present and at our location in different directions $\hat{n}$, so that we expand $\hat{n}$–dependence in terms of spherical harmonics $Y_{lm}(\hat{n})$ as

$$\Delta_T\,(\vec{x}_0, \hat{n}, \tau_0) = \sum_{l,m} a_{lm} Y_{lm}(\hat{n}) \,, \quad (89)$$

where subscript $_0$ denotes evaluation at present.

The angular power spectrum of CMB

- We define the angular power spectrum C_l of the CMB perturbation/anisotropy in terms of a_{lm} as

$$\langle a_{lm} a^*_{l'm'} \rangle = \delta_{ll'} \delta_{mm'} C_l \, . \tag{90}$$

- Due to the statistical homogeneity and isotropy of the perturbation fields, the angular power spectrum C_l does not depend on position and m, and $\langle a_{lm} a^*_{l'm'} \rangle$ vanishes for $l \neq l'$ or $m \neq m'$.

The power spectrum of matter

- The power spectrum of matter at a given redshift can be computed from the transfer function:

$$T(k,z) \equiv \frac{\delta(k,z)}{\delta(0,z)} \frac{\delta(0,z=z_i)}{\delta(k,z=z_i)}\,, \tag{91}$$

  where $\delta(k,z)$ is the density contrast of matter for wave number k and redshift z, and z_i is the redshift at the initial time which usually is the end of inflation.

- The power spectrum of matter is

$$P_k(z) = T^2(k,z)P_i\,, \tag{92}$$

  where P_i is the primordial power spectrum computed from cosmic inflation.

- For the scalar perturbation, the primordial power spectrum can be parameterized as

$$P_i = A_s \left(\frac{k}{k_0}\right)^{1-n_s(k)}\,, \tag{93}$$

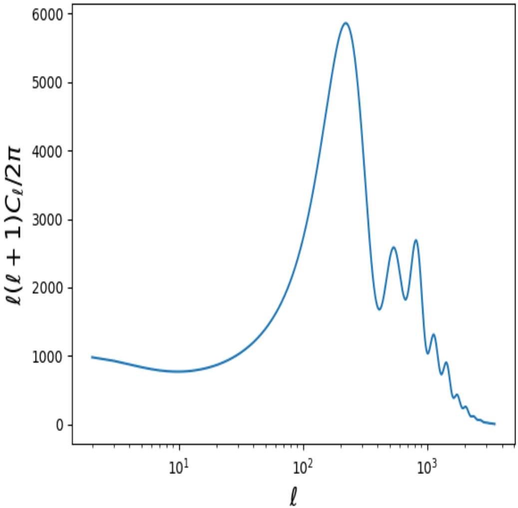

Figure: Angular power spectrum of the temperature perturbations in the Cosmic Microwave Background (CMB).

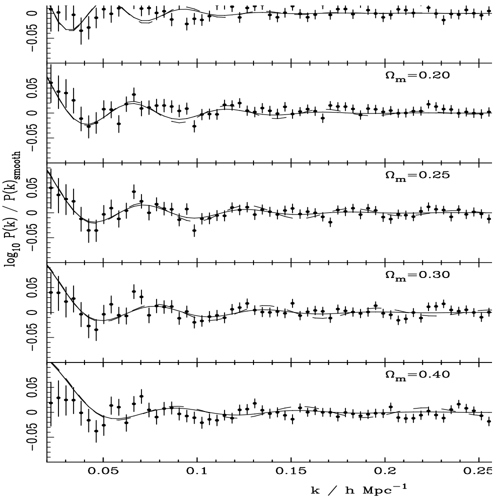

Figure: Oscillation in the matter power spectrum due to baryon oscillation: Baryon Acoustic Oscillation (BAO) (from arXiv:0910.5224). Ω_m^0 varies from 0.1 (top) to 0.4 (bottom).

Tight coupling

- Before recombination the electron number density n_e is large, so that photons and baryons are tightly coupled, with

$$a n_e \sigma_T \equiv \tau_c^{-1} \gg \mathcal{H} \sim \tau^{-1}\,. \tag{94}$$

- In the tight coupling limit, we suppose that $\tau_c/\tau \ll 1$ and $k\tau_c \ll 1$.
- In this limit, we have

$$\dot{\theta}_\gamma = a n_e \sigma_T (\theta_b - \theta_\gamma)\,, \tag{95}$$

$$\dot{\theta}_b = \frac{4\bar{\rho}_\gamma}{3\bar{\rho}_b} a n_e \sigma_T (\theta_\gamma - \theta_b)\,. \tag{96}$$

Tight coupling

- Hence,

$$\dot{\theta}_\gamma - \dot{\theta}_b = -an_e\sigma_T\left(1 + \frac{4\bar{\rho}_\gamma}{3\bar{\rho}_b}\right)(\theta_\gamma - \theta_b)\,, \qquad (97)$$

  yielding $\theta_\gamma \to \theta_b$ exponentially.

- Due to tight coupling, we have

$$\dot{F}_{\gamma\, l} = -an_e\sigma_T F_{\gamma\, l}\,, \quad l \geq 3\,. \qquad (98)$$

  Hence, $F_{\gamma\, l}$ decreases exponentially.

The acoustic oscillation

- The temperature perturbations is related to the average distribution function F as

$$\Delta_T = \frac{1}{4} F_\gamma \, . \quad \Rightarrow \quad \Delta_{T0} = \frac{1}{4} F_{\gamma 0} = \frac{1}{4} \delta_\gamma \, . \qquad (99)$$

- Above the photon diffusion scale, the photons and baryons are tightly coupled.
- At recombination, the diffusion length is much smaller than the Hubble horizon, implying that we consider small scale perturbations.

The acoustic oscillation

- The evolution equation for the temperature perturbations is

$$\ddot{\Delta}_{T0} + \frac{\dot{R}}{1+R}\dot{\Delta}_{T0} + k^2 c_s^2 \Delta_{T0} = \mathcal{F}\,, \tag{100}$$

where c_s^2 is the sound speed of the photon-baryon fluid defined as

$$c_s^2 \equiv \frac{1}{3}\frac{1}{1+R}\,, \tag{101}$$

and the force term is

$$\mathcal{F} \equiv \ddot{\phi} - \frac{k^2}{3}\psi + \frac{\dot{R}}{1+R}\dot{\phi}\,. \tag{102}$$

- The above equation describes the oscillation of the photon-baryon fluid in the gravitational driving forces due to external potentials.

The acoustic oscillation

- The solution can be computed using the Green method as

$$\begin{aligned} & [1+R]^{1/4}\Delta_{T0} \\ = \; & \Delta_{T0}|_I \cos kr_s + \frac{\sqrt{3}}{k}[\dot{\Delta}_{T0} + \frac{1}{4}\dot{R}\Delta_{T0}]_I \sin kr_s \\ & + \frac{\sqrt{3}}{k}\int_{\tau_I}^{\tau} d\tau'[1+R(\tau')]^{3/4} \sin[kr_s(\tau) - kr_s(\tau')]F(\tau')\,, \end{aligned} \tag{103}$$

where a subscript $_I$ denotes evaluation at the initial time.

- Here, r_s is the sound horizon defined as

$$r_s \equiv \int_0^{\tau} c_s d\tau'\,. \tag{104}$$

The acoustic oscillation

- For an adiabatic perturbations, we have

$$\Delta_{T0} \propto \sin(kr_s) \tag{105}$$

- Hence, Δ_{T0} has peaks at

$$kr_s = \frac{2n+1}{2}\pi\,, \quad n \in \{0, 1, 2, \ldots\}\,. \tag{106}$$

Perturbations on large scales

- On large scales the effect of the tight coupling is negligible, so that the evolution equations for radiation (photon + neutrino) and matter (CDM + baryon) can be read from Eqs. (84) and (85) as

$$\dot{\delta}_r = -\frac{4}{3}\theta_r + 4\dot{\phi}\,, \quad \dot{\theta}_r = k^2\frac{1}{4}\delta_r + k^2\psi\,, \tag{107}$$

$$\dot{\delta}_m = 3\phi' - \theta_m\,, \quad \dot{\theta}_m = -\mathcal{H}\theta_m + k^2\psi\,, \tag{108}$$

where a subscript $_r$ denotes radiation while a subscript $_b$ denotes matter.

Perturbations on large scales

- To solve the previous evolution equations, we use the perturbed Einstein equations to write ψ and ϕ in terms of the density contrast and velocity perturbations.
- Adiabatic condition:

$$\delta_m = \frac{1}{4}\delta_r \,. \tag{109}$$

- We get

$$\psi = -\frac{1}{2}\delta_r = \text{constant}\,, \quad \rightarrow \quad \Delta_{T0} = -\frac{1}{2}\psi\,, \tag{110}$$

which describe the temperature perturbations due to the metric perturbations on the last scattering surface. This is an ordinary Sachs-Wolfe effect.

Perturbations on large scales

- When the universe evolves through the matter-radiation equality the metric perturbation on large scales changes.
- This gives the contribution to the ordinary Sachs-Wolfe effect, such that

$$[\Delta_{T0} + \psi] = \frac{1}{3}\psi, \tag{111}$$

which is the Sachs-Wolfe effect.

free streaming

- After recombination, photons decouple from baryons and consequently freely propagate (free streaming) from the last scattering surface to observers at present.
- for $l \geq 2$, the EOM for photon multipoles takes the form of the recursion relation of the spherical Bessel function $j_l(x)$,

$$\begin{aligned} \frac{d}{dx} j_l(x) &= \frac{l}{2l+1} j_{l-1}(x) - \frac{l+1}{2l+1} j_{l+1}(x) \,, \quad (112) \\ \Rightarrow \frac{d}{d\tau} j_l(k\tau) &= \frac{l}{2l+1} k j_{l-1}(k\tau) - \frac{l+1}{2l+1} k j_{l+1}(k\tau) \,. \end{aligned}$$

free streaming

- The EOMs for monopole and dipole of photon perturbations can be written in the form of the recursion relation of the spherical Bessel function as follows:

$$\dot{\Delta}_{T0} = -k\Delta_{T1} + \underbrace{\dot{\phi}}_{\text{extra term}} , \tag{113}$$

$$\dot{\Delta}_{T1} = \frac{k}{3}\Delta_{T0} - \frac{2k}{3}\Delta_{T2} + \underbrace{\frac{k}{3}\psi}_{\text{extra term}} . \tag{114}$$

- The extra term in Eq. (114) can be eliminated if Δ_{T0} is redefined as

$$\tilde{\Delta}_{T0} \equiv \Delta_{T0} + \psi . \tag{115}$$

- Hence,

$$\dot{\tilde{\Delta}}_{T0} = -k\Delta_{T1} + \dot{\phi} + \dot{\psi} , \quad \dot{\Delta}_{T1} = \frac{k}{3}\tilde{\Delta}_{T0} - \frac{2k}{3}\Delta_{T2} . \tag{116}$$

free streaming

- The evolution equations of the photon perturbations can be written for all multipoles as

$$\frac{d}{d\tau}\Delta_{Tl} = \frac{l}{2l+1}k\Delta_{T\,(l-1)} - \frac{l+1}{2l+1}k\Delta_{T\,(l+1)} + \delta_{l0}\left(\dot{\phi}+\dot{\psi}\right)\ . \tag{117}$$

- Let us define

$$X \equiv \int_{\tau_*}^{\tau} d\tau' \left[\dot{\psi}(\tau') + \dot{\phi}(\tau')\right] j_l\left(k\tau - k\tau'\right)\ , \quad Y_l \equiv \Delta_{T\,l} + \delta_{l0}X\,, \tag{118}$$

where τ_* denotes the starting time.

- Hence, Eq. (117) becomes

$$\frac{d}{d\tau}Y_l = \frac{l}{2l+1}kY_{l-1} - \frac{l+1}{2l+1}kY_{l+1}\,. \tag{119}$$

free streaming

- Choosing the starting time of the integration to be $\tau_* = \tau_s$, where τ_s is the conformal time at the Last scattering surface, the solution for the equation in the previous slide is

$$\Delta_{T\,l}(\tau, k) = \underbrace{[\Delta_{T0} + \psi]}_{\text{SW effect}}(\tau_s, k)\, j_l\,(k(\tau - \tau_s)) + \int_{\tau_s}^{\tau} d\tau' \left[\dot{\psi}(\tau') + \dot{\phi}(\tau')\right] j_l\,(k\tau - k\tau')\ , \quad (120)$$

where a subscript $_s$ denotes evaluation on the Last scattering surface, the second term on the RHS of this equation is the contribution to the temperature perturbations from the time-dependence of the metric perturbations, and this contribution is the integrated Sachs-Wolfe (ISW) effect arising during free streaming.

free streaming

- The free streaming can distribute oscillation of the temperature perturbations in photon-baryon fluid on the Last scattering surface to higher multipoles.
- $$\Delta_{T\,l}\left(\tau_0, k\right) = \Delta_{T0\,,osc}\left(\tau_s, k\right) j_l\left(k\tau_*\right)\ , \qquad (121)$$

  where $\tau_* \equiv \tau_0 - \tau_s$.
- For the adiabatic case, $\Delta_{T0\,,osc}\left(\tau_s, k\right)$ has peaks at

$$k = \frac{n\pi}{2r_s}\ , \quad n \in \{1, 3, 5, ...\}\ . \qquad (122)$$

free streaming

- According to the maximum of the spherical Bessel function $j_l(k\tau_*)$ at $l \simeq k\tau_*$, the maximum of Δ^2_{T0osc} observed at the present occurs around

$$l = \frac{n\pi}{2r_s}\tau_* = nl_A\,, \quad n \in \{1, 3, 5, ...\}\,. \tag{123}$$

where l_A is a characteristic acoustic index.

Line of Sight Integration

- The reference of this topic is [astro-ph/9603033]. We define differential optical depth for Thomson scattering as $\dot{\kappa} = a n_e \sigma_T$.
- When the polarization is included, the Boltzmann equation for photon perturbations becomes

$$\dot{\Delta}_T^{(S)} + ik\mu\Delta_T^{(S)} = \phi' - ik\mu\psi$$
$$+\dot{\kappa}\left[-\Delta_T^{(S)} + \Delta_{T0}^{(S)} + i\mu v_b + \frac{1}{2}P_2(\mu)\Pi\right] \,, \quad (124)$$

where $\Delta_T^{(S)}$ is the scalar mode of the temperature perturbation.

- Here,

$$\Pi = \Delta_{T2}^{(S)} + \Delta_{P2}^{(S)} + \Delta_{P0}^{(S)} \,, \quad (125)$$

where $\Delta_P^{(S)}$ is the scalar mode of polarization in temperature perturbation.

Line of Sight Integration

- The Boltzmann equation can be written in the integral form as

$$\Delta_T^{(S)} = \int_0^{\tau_0} d\tau \underbrace{e^{ik\mu(\tau-\tau_0)}}_{\text{angular dependent}} S_T^{(S)}(k,\tau)\,. \tag{126}$$

- For the temperature perturbation, the source term is

$$S_T^{(S)}(k,\tau) = g\left(\Delta_{T0} + \psi - \frac{v_b'}{k} - \frac{\Pi}{4} - \frac{3\Pi''}{4k^2}\right) \tag{127}$$

$$+ \mathrm{e}^{-\kappa}(\phi' + \psi') - g'\left(\frac{v_b}{k} + \frac{3\Pi'}{4k^2}\right) - \frac{3g''\Pi}{4k^2}\,. \tag{128}$$

Line of Sight Integration

- We use the decomposition

$$e^{ik\mu(\tau-\tau_0)} = \sum_{l=0}^{\infty}(2l+1)(-i)^l j_l(k(\tau_0-\tau))P_l(\mu)\,. \quad (129)$$

- We finally get

$$\Delta^{(S)}_{(T,P)l}(k,\tau=\tau_0) = \int_0^{\tau_0} S^{(S)}_{T,P}(k,\tau) j_l[k(\tau_0-\tau)]d\tau, \quad (130)$$

- The advantage of (130) is the decomposition of $\Delta^{(S)}_{(T,P)l}$ in to $S^{(S)}_{T,P}$, which does not depend on the multipole moment l and a geometrical term j_l, which does not depend on the particular cosmological model.

CLASS

- input.c $\Rightarrow$ precompute Bessel function, read cosmological parameters, tune initial conditions
- background.c $\Rightarrow$ solve evolution equations for the background universe
- thermodynamics.c $\Rightarrow$ compute evolution of number density of electron
- perturbations.c $\Rightarrow$ compute evolution of the perturbations in all species
- compute CMB angular power spectrra and matter power spectra

Basic of 3+1 decomposition

- To use package xTensor in Mathematica to compute evolution equations for the perturbations, it is convenient to perform the calculation in 3+1 formalism. The reference is gr-qc/0703035.
- In vector analysis, we can define a two-dimensional plane in three-dimensional space using the unit vector $\vec{n}$ that is normal to the plane.
- We can decompose any vector in three-dimensional space to two parts.
- The first is the projection of the vector along $\vec{n}$, and the second is the projection on the plane.

Basic of 3+1 decomposition

- A vector $\vec{A}$ can be projected along $\vec{n}$ using a scalar product $A_n = \vec{A} \cdot \vec{n}$
- Hence, the vector $\vec{A}_p \equiv \vec{A} - \vec{n}A_n$ is a vector that is normal to $\vec{n}$, i.e., lies on the plane.
- We can define the projection operater, $\mathbf{P} \equiv \mathbf{I} - \vec{n} \otimes \vec{n}$, to project any vector on the plane defined by normal vector $\vec{n}$, where $\mathbf{I}$ is the identity matrix and $\otimes$ denotes the tensor product.

Basic of 3+1 decomposition

- In four-dimensional spacetime, the three-dimensional hypersurface can be defined using a one-form n_μ that is normal to any vectors lie on the hypersurface.
- The hypersurfaces are space-like if their normal one-form is time-like, while they are time-like if the normal one-form is space-like.
- We use the unit time-like n_μ to define the space like hypersurfaces.
- Hence, the projection operator is defined as

$$P_\alpha^\beta = \delta_\alpha^\beta + n_\alpha n^\beta \,. \tag{131}$$

Basic of 3+1 decomposition

- It is easy to show that $P_\alpha^\beta n^\alpha = 0$ because n^α is normal to the hypersurfaces.
- Tensor $T_{\mu\nu}$ is on hypersurfaces

$$T_{\mu\nu} = P_\mu^\alpha T_{\alpha\nu} \,. \tag{132}$$

- The projection of the metric tensor $g_{\mu\nu}$ on the hypersurfaces is

$$g_{\mu\nu} P_\alpha^\mu P_\beta^\nu = g_{\alpha\beta + n_\alpha n_\beta} \equiv h_{\alpha\beta} \,, \tag{133}$$

where $h_{\alpha\beta}$ is the metric tensor of the space-like hypersurface defined by n_μ.

Basic of 3+1 decomposition

- The metric tensor can be written in the form of the Arnowitt–Deser–Misner (ADM) metric as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + h_{ij}(N^i dt + dx^i)(N^j dt + dx^j)\,, \tag{134}$$

where N is a Lapse function and n^i (defined on the hypersurface) is a shift vector.

- The Latin indices denote the indices of the quantities on the hypersurface, which are raised or lowered by the metric h_{ij}.
- The metric h_{ij} can be used to compute the intrinsic curvature of the hypersurface as

$$^{(3)}R_{ij} = \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k_{jk} + \Gamma^k_{ij}\Gamma^l_{kl} - \Gamma^l_{ik}\Gamma^k_{jl}\,, \tag{135}$$

where $^{(3)}R_{ij}$ is the Ricci tensor of the hypersurface.

Basic of 3+1 decomposition

- The Christoffel symbol on the hypersurface:

$$\Gamma^k_{ij} = \frac{1}{2} h^{kl} \left(\partial_i h_[jl] + \partial_j h_{il} - \partial_l h_{ij} \right) . \tag{136}$$

- The extrinsic curvature of the hypersurface can be computed from

$$K_{ij} = \frac{1}{2N} \left(\dot{h}_{ij} - D_i n_j - D_j n_i \right) . \tag{137}$$

- The Ricci scalar of the four-dimensional spacetime can be expressed as

$$R =^{(3)} R + K_{ij} K^{ij} - K^2 , \tag{138}$$

where ${}^{(3)}R = h^{ij\,(3)} R_{ij}$ and $K = h^{ij} K_{ij}$.

Basic of 3+1 decomposition

- The component 0-0 of the Einstein equation can be obtained by projecting both indices along n^μ:

$$G_{\mu\nu} n^\mu n^\nu = \frac{1}{2} {}^{(3)}R + \frac{1}{2} K^2 - \frac{1}{2} K_{ij} K^{ij} = T_{\mu\nu} n^\mu n^\nu \,, u \quad (139)$$

  where we have set $8\pi G = 1$.

- The component 0-i of the Einstein equation can be obtained by projecting one index along n^μ and another index on the hypersurface:

$$P^\mu_\alpha n^\nu G_{\mu\nu} = P^\mu_\alpha n^\nu R_{\mu\nu} = D_i K - D_j K^j_i = P^\mu_\alpha n^\nu T_{\mu\nu} \,, \quad (140)$$

  where D_i is the covariant derivative compatible with h_{ij}.

- In terms of the Lapse function and the shift vector, we have

$$n^\mu = \left(\frac{1}{N}, -\frac{N^i}{N} \right) \,, \quad n_\mu = (-N, 0) \,. \quad (141)$$

Perturbations

We consider the Newtonian gauge:

$$g_{0i} = N_i = 0\,, \tag{142}$$

$$g_{00} = -N^2 = -a^2\left(1 + 2\psi\right)\,, \tag{143}$$

$$g_{ij} = h_{ij} = a^2\left(1 - 2\phi\right)\delta_{ij}\,, \tag{144}$$

$$n^\mu = \frac{1}{a}\left(\frac{1}{(1 + 2\delta N)}, 0\right)\,. \tag{145}$$