Green Transformation and Fiscal Policy in a Two-Sector OLG Model with Production Externalities*

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SSRN Working-Paper ID 5169780

March, 2025

Abstract

This article studies the decarbonisation of an overlapping-generations economy with a polluting (brown) and a non-polluting (green) production sector that produce a single good. Agents derive utility from consumption and disutility from pollution, which is an intergenerational externality. We characterise the optimal allocation of a social planner and establish the existence of at least one modified golden-rule steady state. Depending on the economy's fundamentals, multiple modified golden-rule steady states with different pollution levels may coexist. We construct a balanced-budget fiscal policy consisting of an emissions tax and intergenerational transfers that implements the optimal allocation. To attain the social optimum, the tax receipts must be released to compensate agents for the reduction in factor incomes caused by the emissions tax. The optimal emissions tax rate is the sum of the discounted marginal future damages.

Keywords: Overlapping generations, two-sector growth models, production externalities, green transformation, climate policy, golden rules

JEL Classification: D61, D62, E62, H23, O41, P28

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[*] **Acknowledgements.** We would like to thank Marten Hillebrand, Daniel Heyen, Tom Rauber, and the participants of research seminars at the University of Kaiserslautern-Landau and the University of Freiburg for helpful comments and suggestions. We also benefited from numerous discussions at the International Workshop *"Optimal Timing and Control to Eradicate Firms" Polluting Activities: The Roles of Fiscal and Monetary Policies"* in Urbino, Italy, 2024.

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1 Introduction

The green transformation of large economies is a ubiquitous topic in world politics. Its urgency is driven by the increasingly severe effects of anthropogenic climate change and its tremendous economic costs, e.g., see Stern (2007). Since the industrial and energy sectors account for more than half of all greenhouse-gas emissions (Lee et al., 2023; Friedlingstein et al., 2022), a central challenge for policymakers is to incentivise firms to adopt less emission-intensive technologies. Green technologies are readily available for most industries and are becoming more and more economically viable (Lee et al., 2023). Emissions reduction enhances the competitiveness of firms by mitigating the financial burden of carbon pricing and by aligning with consumers' growing concern for the environment.

Mitigating climate change involves an intricate intergenerational challenge. Although today's emissions jeopardise the welfare of all future generations, the decarbonisation of an economy is an incremental process that must be harmonised with economic prosperity. The abatement costs borne by present generations have to be weighed against the benefits for future generations. Numerous contributions such as Stephan, Müller-Fürstenberger, and Previdoli (1997); Howarth (2000); or Schneider, Traeger, and Winkler (2012) highlight the suitability of overlapping-generations (OLG) models for addressing intergenerational aspects of climate economics. OLG models complement the more traditional Ramsey-type growth models, which feature an infinitely-lived representative agent (ILA). While it is well known that certain assumptions on altruism can generate an *'observational equivalence'* between OLG and ILA models, e.g., see Barro (1974) and Schneider et al. (2012), the extent to which this equivalence applies when intergenerational externalities such as climate change matter is unclear. The current debate on fighting climate change demonstrates quite clearly that consumption and production decisions are driven primarily by individual desires rather than altruistic motives for the benefit of future generations.

Since the pioneering contributions by Howarth and Norgaard (1992) and John and Pecchenino (1994), a substantial body of literature has incorporated pollution externalities into OLG models. The origins and effects of these externalities are modelled in various ways. They can be caused by production (Howarth, 1998; Marini & Scaramozzino, 1995), consumption (John & Pecchenino, 1994; John, Pecchenino, Schimmelpfennig, & Schreft, 1995; Ono, 1996), labour (Andersen, Bhattacharya, & Liu, 2020), energy use (Howarth & Norgaard, 1992), or the extraction of natural resources (Mourmouras, 1991). In the models by Howarth and Norgaard (1992); Andersen et al. (2020), and Goussebaïle (2024), pollution reduces future output. In Gutiérrez (2008), it reduces consumption, whereas in the model by John et al. (1995), pollution reduces welfare directly because agents have environmental preferences.

Despite the abundance of OLG models in climate economics, analytically tractable multi-sector models are rare. Rausch and Yonezawa (2023) compare the intergenerational welfare effects of technology policies with those of carbon pricing in a setting with a green and a brown intermediate-goods sector. They find that a green technology policy, unlike carbon pricing, may act as a capital subsidy that disproportionately benefits current generations at the expense of future generations. The model in Dao and Edenhofer (2018), which, similar to Rausch and Yonezawa (2023) has two heterogeneous intermediate-goods sectors, generates poverty-environment traps, i.e., steady states with a poor environmental quality and a low capital stock. The authors demonstrate that the optimal allocation can be decentralised by taxing emissions and capital income. Poverty-environment traps also occur in the multi-sector OLG model by Ikefuji and Horii (2007). In a setting with a resource and a production sector, Karp and Rezai (2014) investigate the extent to which environmental policies can achieve a Pareto improvement by changing asset prices. However, since the capital stock is given exogenously, the model is not designed to study climate policies. More elaborate models with multiple sectors require a numerical analysis, e.g., see Kotlikoff, Kubler, Polbin, Sachs, and Scheidegger (2021).

This article examines the industrial decarbonisation in an analytically tractable two-sector OLG model. In the spirit of Galor (1992), Diamond's (1965) classical framework is extended to two production sectors, a polluting brown and a non-polluting green sector. Both sectors compete for capital and labour and produce the same good, but pollution degrades the environment. Since agents have preferences over consumption and the quality of the environment, pollution has a negative effect on the welfare of all future generations. Using a novel parameterisation of the production-possibility frontier, our approach naturally incorporates capital-intensity reversals. In contrast to existing multi-sector models, an a-priori assumption on relative capital intensities, as for example in Rausch and Yonezawa (2023), is not needed. In particular, our framework allows for boundary factor allocations in which production is entirely green or entirely brown. To the best of our knowledge, our article develops the first analytically tractable OLG model with pollution externalities that includes two final sectors.

Without emission pricing, the market economy will not internalise the social cost of pollution caused by the brown sector. To attain the social optimum, the brown output must be reduced and agents must forgo consumption. Therefore, the welfare maximum will, in general, deviate from the consumption maximum. This feature distinguishes our model from the standard OLG model. Adopting the perspective of a social planner, we investigate the trade-off between consumption and pollution abatement. We show that there exists a uniquely determined optimal allocation and at least one welfare-maximising steady state, i.e., a *modified golden-rule steady state*. Depending on the sector-specific productivity, discount factors, and agents' valuation of the externality, the modified golden-rule steady state may be green, brown, or mixed. It turns out that the steady-state pollution levels depend on the relationship between the marginal rate of transformation and the marginal rate of substitution between consumption and pollution. Since all three types of steady states may coexist, our model may help to explain the existence of economies with high and with low pollution levels.

The main contribution of this article concerns the decentralisation of the optimal allocation, which has important policy implications. We demonstrate that the optimal allocation can be implemented in the market economy by means of a balanced-budget fiscal policy. This policy consists of an emissions tax that internalises the pollution externality by shifting production factors to the green sector, combined with intergenerational transfers. The emissions tax stipulates the wage-rental ratio and thus determines the production plan and the emissions level. Since the emissions tax reduces the factor incomes by distorting factor prices, the tax receipts must be released to agents in order to offset welfare losses. The intergenerational transfers enable the government to distribute the tax receipts among the two generations and implement the optimal capital accumulation path. This resolves the dynamic inefficiency inherent to all OLG models. In line with the existing literature, e.g., see Jaimes (2023), we find that the optimal emissions tax rate is equal to the sum of the discounted marginal damages incurred by all future generations. Our analysis extends that of Dao and Davila (2014) who consider a one-sector OLG model with a production externality and environmental preferences, demonstrating how a golden-rule steady state can be implemented by means of various transfer schemes.

This article is organised as follows. The next section introduces the assumptions on technologies and preferences and defines the economy's production-possibility frontier. In Section 3, we adopt the perspective of a benevolent social planner and establish the optimal allocation and modified golden-rule steady states. The implementation of the optimal allocation in a market economy through fiscal policy is addressed in Section 4. Section 5 concludes. All proofs are collected in an appendix.

2 Model

We consider an overlapping-generations model with discrete time $t = 0, 1, \ldots, \infty$ and two-period lived agents. At the beginning of each period $t \geq 0$, a new generation consisting of a continuum of homogeneous agents with mass $N_t \in \mathbb{R}_{++}$ is born. The population grows exponentially at the rate $n \geq 0$, so that $N_t = (1 + n)^t N_0$.

2.1 Production

A single consumption-and-investment good is produced from the factors real capital $K_t \geq 0$ and labour $L_t \geq 0$ by a non-polluting *'green'* and a polluting *'brown'* sector, indexed by $j = g, b$, respectively. Each sector comprises a continuum of homogeneous, price-taking firms. The brown sector's production externality is explained in Section 2.3 below. The workforce consists of young agents who supply one unit of labour inelastically, so that $L_t = N_t$. Old agents are retired and consume the capital income generated by renting capital to firms. Capital and labour can move frictionlessly between the sectors and are paid their marginal products. The output good serves as the numeraire.

The production function of the representative firm in sector $j = g, b$ is

$$F_j : \mathbb{R}^2_+ \to \mathbb{R}_+, \qquad Y^j = F_j(K^j, L^j),$$

where $Y^j \geq 0$ is the sector-specific output and $K^j, L^j \geq 0$ are the sector-specific capital and labour inputs. Assuming perfect substitutability, the economy's total output of the good is $Y = Y^g + Y^b$. The technology in either sector has constant returns to scale, meaning that both production functions F_g and F_b are linear homogeneous. Their respective intensive forms $f_j : \mathbb{R}_+ \to \mathbb{R}_+$ obtain by setting

$$f_j(k^j) := F_j\big(\tfrac{K^j}{L^j}, 1\big), \quad j = g, b,$$

where $k^j = \frac{K^j}{L^j}$ is the sector-specific capital-labour ratio. The marginal product of labour $w_j : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$w_j(k^j) := f_j(k^j) - f_j'(k^j)k^j, \quad j = g, b. \tag{2.1}$$

The marginal rate of technical substitution (MRTS) between capital and labour is given by the function $\Omega_j : \mathbb{R}_+ \to \mathbb{R}_+$, defined by

$$\Omega_j(k^j) := \frac{w_j(k^j)}{f_j'(k^j)}, \quad j = g, b. \tag{2.2}$$

Given the output price $p_t^j > 0$, the wage rate $w_t > 0$, and the capital-rental rate $r_t > 0$, the profit-maximization problem of the representative firm $j = g, b$ is

$$\max_{K^j, L^j \geq 0} \; p_t^j F_j(K^j, L^j) - r_t K^j - w_t L^j. \tag{2.3}$$

The first-order conditions for the profit-maximising factor inputs (K_t^j, L_t^j) are

$$\begin{aligned} (i) \qquad & p_t^j \frac{\partial F_j}{\partial K}(K_t^j, L_t^j) \;=\; p_t^j f_j'(k_t^j) \;\stackrel{!}{=}\; r_t \\ (ii) \qquad & p_t^j \frac{\partial F_j}{\partial L}(K_t^j, L_t^j) \;=\; p_t^j w_j(k_t^j) \;\stackrel{!}{=}\; w_t, \end{aligned} \tag{2.4}$$

showing that due to linear homogeneity, only the capital-labour ratio k_t^j is well defined by the first-order conditions (2.4). Given any wage-rental ratio $\omega_t = \frac{w_t}{r_t}$, these reduce to

$$\Omega_j(k_t^j) \stackrel{!}{=} \omega_t. \tag{2.5}$$

Denote the social discount factor by $0 < \gamma < 1$ and the capital depreciation rate by $0 < \delta \leq 1$, then we can introduce the following assumptions on the production technologies.

Assumption 1 (Technology).
The two production sectors $j = g, b$ are characterised by the following properties.

(i) The production functions $f_j : \mathbb{R}_+ \to \mathbb{R}_+$ are twice continuously differentiable, strictly increasing, $f_j' > 0$, strictly concave, $f_j'' < 0$, and satisfy

$$(a) \quad \lim_{k \to 0} f_j'(k) > \tfrac{1+n}{\gamma} - 1 + \delta \quad \text{and} \quad (b) \quad \lim_{k \to \infty} f_j'(k) = 0. \tag{2.6}$$

Moreover, capital is an essential production factor, that is, $f_j(0) = 0$.

(ii) The MRTS functions $\Omega_j : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the boundary conditions

$$\lim_{k\to 0} \Omega_j(k) = 0 \quad \textit{and} \quad \lim_{k\to\infty} \Omega_j(k) = \infty. \tag{2.7}$$

Since Assumption 1 (i) implies that the MRTS functions Ω_j, $j = g, b$, are strictly increasing, the boundary conditions (2.7) ensure that the first-order condition (2.5) admits a unique solution k_t^j for any given wage-rental ratio $\omega_t \geq 0$. Condition (2.6) (a) is weaker than the Inada condition $\lim_{k\to 0} f_j'(k) = \infty$ and thus allows for CES production functions. Condition (2.6) (b) is needed later to rule out unbounded economic growth.

2.2 Production-possibility frontier

Assumption 1 implies that there exist *relative factor-demand functions* $\kappa_j : \mathbb{R}_+ \to \mathbb{R}_+$, $j = g, b$, such that for any wage-rental ratio $\omega_t \geq 0$, the first-order condition (2.5) is satisfied, i.e.,

$$\Omega_j(\kappa_j(\omega_t)) = \omega_t. \tag{2.8}$$

These functions stipulate the sector-specific capital-labour ratios, so that $k_t^g = \kappa_g(\omega_t)$ and $k_t^b = \kappa_b(\omega_t)$. The corresponding labour shares $l_t^j = \frac{L_t^j}{L_t}$, $j = g, b$, are obtained as follows. For each economy-wide capital-labour ratio $k_t \geq 0$, set

$$\Omega_{\min}(k_t) := \min\{\Omega_g(k_t), \Omega_b(k_t)\} \quad \text{and} \quad \Omega_{\max}(k_t) := \max\{\Omega_g(k_t), \Omega_b(k_t)\}. \tag{2.9}$$

For each $k_t > 0$ with $\Omega_g(k_t) \neq \Omega_b(k_t)$, the *labour-share functions*

$$\ell_j(k_t, \cdot) : [\Omega_{\min}(k_t), \Omega_{\max}(k_t)] \to [0, 1], \quad j = g, b,$$

are then defined by setting

$$l_t^g = \ell_g(k_t, \omega_t) := \frac{\kappa_b(\omega_t) - k_t}{\kappa_b(\omega_t) - \kappa_g(\omega_t)} \quad \text{and} \quad l_t^b = \ell_b(k_t, \omega_t) := \frac{k_t - \kappa_g(\omega_t)}{\kappa_b(\omega_t) - \kappa_g(\omega_t)}, \tag{2.10}$$

respectively. For each $k_t > 0$ with $\Omega_g(k_t) \neq \Omega_b(k_t)$ and each wage-rental ratio $\omega_t \in [\Omega_{\min}(k_t), \Omega_{\max}(k_t)]$, the list $(k_t^g, k_t^b, l_t^g, l_t^b) \geq 0$ defined by (2.8) and (2.10) is a *feasible factor allocation* as it solves the market-clearing conditions in the capital and the labour market

$$(a) \quad k_t = l_t^g k_t^g + l_t^b k_t^b \quad \text{and} \quad (b) \quad 1 = l_t^g + l_t^b. \tag{2.11}$$

Since, by (2.8), $\Omega_g(k_t^g) = \Omega_b(k_t^b)$ is also satisfied, this factor allocation is *(Pareto-)efficient* in the sense that the output of a sector cannot be raised without lowering the output of the other sector.[1] In particular,

$$\ell_g(k_t, \Omega_b(k_t)) = \ell_b(k_t, \Omega_g(k_t)) = 0 \quad \text{and} \quad \ell_g(k_t, \Omega_g(k_t)) = \ell_b(k_t, \Omega_b(k_t)) = 1,$$

[1] In the following, we will omit the word *'Pareto'* in order to avoid confusion with the notion of Pareto-optimality of the whole economy, introduced below.

so that boundary factor allocations in which only one sector is producing are included.

For each $k_t \geq 0$ with $\Omega_g(k_t) = \Omega_b(k_t)$, the market-clearing conditions (2.11) are satisfied with $k_t^g = k_t^b = k_t$ and any feasible allocation of labour.

Our approach naturally incorporates *capital-intensity reversals*. The marginal rates of technical substitution determine whether the green or the brown sector is more capital-intensive. If $\Omega_b(k_t) < \Omega_g(k_t)$, then

$$\kappa_g(\omega) < k_t < \kappa_b(\omega) \quad \text{for all} \;\; \omega \in \big(\Omega_b(k_t), \Omega_g(k_t)\big),$$

and vice versa, if $\Omega_g(k_t) < \Omega_b(k_t)$, then

$$\kappa_b(\omega) < k_t < \kappa_g(\omega) \quad \text{for all} \;\; \omega \in \big(\Omega_g(k_t), \Omega_b(k_t)\big).$$

The production-possibility frontier may now be described as follows. For any $k_t > 0$ with $\Omega_g(k_t) \neq \Omega_b(k_t)$ and any $\omega_t \in \big[\Omega_{\min}(k_t), \Omega_{\max}(k_t)\big]$, the per-capita output of sector $j = g, b$ is stipulated by the function

$$y_t^j = \mathsf{y}_j(k_t, \omega_t) := \ell_j(k_t, \omega_t) f_j(\kappa_j(\omega_t)). \tag{2.12}$$

In the non-generic case $\Omega_g(k_t) = \Omega_b(k_t)$, the per-capita outputs are

$$y_t^g = l_t^g f_g(k_t) \quad \text{and} \quad y_t^b = (1 - l_t^g) f_b(k_t), \quad l_t^g \in [0, 1]. \tag{2.13}$$

The production plan (y_t^b, y_t^g) is called *efficient* because it is produced with an efficient factor allocation. Since for any fixed $k_t > 0$, the output functions (2.12) and (2.13) are invertible[2], each efficient production plan (y_t^b, y_t^g) defines a uniquely determined efficient factor allocation. As a consequence, the function $T : \mathcal{D} \to \mathbb{R}_+$, defined by

$$y_t^g = T(k_t, y_t^b) := \begin{cases} \mathsf{y}_g\big(k_t, \mathsf{y}_b^{-1}(k_t, y_t^b)\big) & \text{if } \Omega_g(k_t) \neq \Omega_b(k_t) \\ f_g(k_t) - \frac{f_g(k_t)}{f_b(k_t)} y_t^b & \text{if } \Omega_g(k_t) = \Omega_b(k_t) \end{cases}, \tag{2.14}$$

where

$$\mathcal{D} := \Big\{ (k, y^b) \in \mathbb{R}_{++} \times \mathbb{R}_+ \;\big|\; y^b \leq f_b(k) \Big\},$$

is well defined. For any $k_t > 0$, the production plan $(y_t^b, T(k_t, y_t^b))$ is efficient.

The following lemma presents properties of the function T that are essential for our results.

Lemma 1 (Concavity of T, Ritschel & Wenzelburger, 2024).
Under the hypotheses of Assumption 1, the map $T : \mathcal{D} \to \mathbb{R}_+$ is concave. For each

[2]The well-known Stolper-Samuelson theorem states that the functions (2.12) are monotonic w.r.t. ω, so that the inverse functions $\mathsf{y}_j^{-1}(k_t, \cdot) : [0, f_j(k_t)] \to \big[\Omega_{\min}(k_t), \Omega_{\max}(k_t)\big],\ j = g, b$, are well defined, satisfying $\mathsf{y}_j^{-1}\big(k_t, \mathsf{y}_j(k_t, \omega_t)\big) = \omega_t$. Formally, the invertibility along with the invertibility of the labour-share functions is shown in Ritschel and Wenzelburger (2024).

$(k_t, y_t^b) \in \mathcal{D}$, *its partial derivatives are*

$$(i) \quad \frac{\partial T}{\partial y^b}(k_t, y_t^b) = -\frac{f_g'(\kappa_g(\omega_t))}{f_b'(\kappa_b(\omega_t))} \quad \textit{and} \quad (ii) \quad \frac{\partial T}{\partial k}(k_t, y_t^b) = f_g'(\kappa_g(\omega_t)),$$

where

$$\omega_t = \Omega(k_t, y_t^b) := \begin{cases} \mathsf{y}_b^{-1}(k_t, y_t^b) & \textit{if} \;\; \Omega_g(k_t) \neq \Omega_b(k_t) \\ \Omega_b(k_t) & \textit{if} \;\; \Omega_g(k_t) = \Omega_b(k_t) \end{cases}.$$

For any fixed $k_t > 0$, the curve

$$T(k_t, \cdot) : \big[0, f_b(k_t)\big] \to \big[0, f_g(k_t)\big], \qquad y^b \mapsto T(k_t, y^b), \tag{2.15}$$

defines the production-possibility frontier pertaining to the capital-labour ratio k_t. The total output per capita plus depreciated capital produced with the production plan $(y_t^b, T(k_t, y_t^b))$ is given by the function $f : \mathcal{D} \to \mathbb{R}_+$, defined by

$$f(k_t, y_t^b) := y_t^b + T(k_t, y_t^b) + (1 - \delta)k_t. \tag{2.16}$$

Lemma 1 implies that the function f is concave, but not necessarily strictly concave. In particular, for any given capital-labour ratio $k_t > 0$, the production-possibility frontier (2.15) is concave.

Lemma 1 suggests to define the function $\varrho : \mathbb{R}_+ \to \mathbb{R}_+$ by setting

$$\varrho(\omega) := \frac{f_g'(\kappa_g(\omega))}{f_b'(\kappa_b(\omega))}. \tag{2.17}$$

The marginal rate of transformation corresponding to any production plan $(y_t^b, T(k_t, y_t^b))$ then is $\varrho(\Omega(k_t, y_t^b))$. Two observations, formally established in Ritschel and Wenzelburger (2024), are important and portrayed in Figure 1. First, the concavity of T implies that $\varrho(\Omega_g(k_t)) \leq \varrho(\Omega_b(k_t))$ for all $k_t > 0$, independently of the sector-specific capital intensities. Second, the marginal rate of transformation is increasing in the wage-rental ratio if and only if the green sector is more capital-intensive than the brown sector, i.e., $\varrho'(\omega_t) \geq 0 \iff \kappa_g(\omega_t) \geq \kappa_b(\omega_t)$.

2.3 Production externality

To produce one unit of the good, the brown sector emits $\epsilon > 0$ pollution units into the environment. The pollution stock per capita of the young generation *at the beginning* of period t is denoted by $e_t \geq 0$. The emissions per capita generated in period t are $\epsilon\, y_t^b$. The evolution of the per-capita pollution stock over time is then governed by a function $E : \mathbb{R}_+^2 \to \mathbb{R}_+$, defined by

$$e_{t+1} = E(e_t, y_t^b) := \tfrac{1}{1+n}\big[(1 - \zeta)e_t + \epsilon y_t^b\big], \tag{2.18}$$

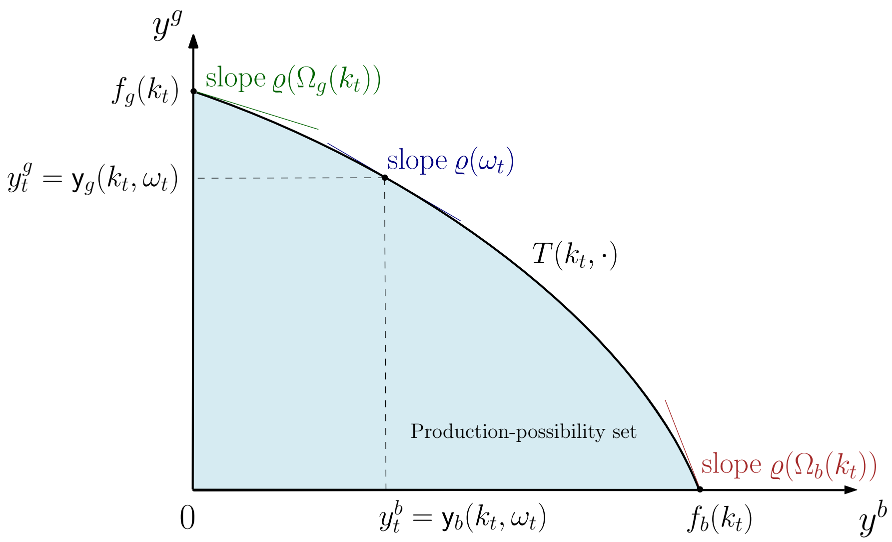

Figure 1: Strictly concave production-possibility frontier; $k_t > 0$ fixed with $\Omega_g(k_t) \neq \Omega_b(k_t)$.

where $0 < \zeta \leq 1$ is the pollution decay rate. The pollution index in period t is

$$z_t = (1-\zeta)e_t + \epsilon y^b_t = (1+n)e_{t+1}. \tag{2.19}$$

It follows from (2.18) that[3]

$$e_{t+1} = \left(\tfrac{1-\zeta}{1+n}\right)^t e_0 + \tfrac{\epsilon}{1+n} \sum_{j=0}^{t} \left(\tfrac{1-\zeta}{1+n}\right)^{t-j} y^b_j. \tag{2.20}$$

Therefore, unless pollution decays fully (i.e., $\zeta = 1$), it diminishes the welfare of all future generations. The production externality thus has both, *intra-* and *intergenerational* effects. Observe, however, that our model abstracts from a direct effect of pollution on production.

2.4 Preferences

Agents have preferences over consumption and the quality of the environment. The preferences are represented by an additive-separable life-cycle utility function $U : \mathbb{R}^4_+ \to \mathbb{R}$, defined by

$$U(c^1_t, c^2_{t+1}, z_t, z_{t+1}) \coloneqq u(c^1_t) - \mu(z_t) + \beta\big[u(c^2_{t+1}) - \mu(z_{t+1})\big], \tag{2.21}$$

where $c^1_t, c^2_{t+1} \geq 0$ denote youthful and old-age consumption, respectively, and $z_t \geq 0$ is the environmental pollution index defined in (2.19). The factor $\beta > 0$ is the agent-specific time-discount factor.

Our assumptions on the preferences are the following.

Assumption 2 (Preferences)**.**

[3] Since the sequence $\{y^b_t\}_{t=0}^{\infty}$ is bounded from above by some $y^b_{\max} > 0$ and $\frac{1-\zeta}{1+n} \in [0, 1)$, the sequence $\{e_t\}_{t=0}^{\infty}$ is bounded from above by $e_{\max} = \frac{\epsilon}{n+\zeta}\, y^b_{\max}$.

(i) The utility function $u : \mathbb{R}_+ \to \mathbb{R}$ is twice continuously differentiable, strictly increasing, $u' > 0$, strictly concave, $u'' < 0$, and satisfies the Inada condition $\lim_{c\to 0} u'(c) = \infty$.

(ii) The damage function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is twice continuously differentiable, strictly increasing, $\mu' > 0$, and convex, $\mu'' \geq 0$.

Assumption 2 implies that youthful and old-age consumption are normal goods, whereas environmental pollution is a *bad*. The total pollution damage incurred by an agent born in period t over his whole lifetime is $\mu(z_t) + \beta\mu(z_{t+1})$.

3 Welfare Analysis

This section examines welfare aspects of the model by taking the perspective of a benevolent social planner who maximises the sum of the discounted welfare levels of all generations. In doing so, the planner takes the intergenerational effects of pollution into account. In particular, he weighs the current generations' utility of consumption against the pollution damages inflicted on future generations.

3.1 Feasible allocations

For any given capital-labour ratio $k_t \geq 0$, the *set of feasible policies* (y_t^b, c_t^1, c_t^2) is defined by

$$Q(k_t) := \Big\{(y^b, c^1, c^2) \in \mathbb{R}_+^3 \mid c^1 + \tfrac{c^2}{1+n} \leq f(k_t, y^b) \text{ and } y^b \leq f_b(k_t)\Big\}. \tag{3.1}$$

Since, by Lemma 1, the production function f is concave, each set $Q(k_t)$ is compact and convex.

In each period $t \geq 0$, the total output $f(k_t, y_t^b)$ must equal total consumption and investments. In per-capita terms, this resource constraint translates into the *capital accumulation law* $A : \mathcal{D} \times \mathbb{R}_+^2 \to \mathbb{R}_+$, defined by

$$k_{t+1} = A(k_t, y_t^b, c_t^1, c_t^2) := \tfrac{1}{1+n}\Big[f(k_t, y_t^b) - \big(c_t^1 + \tfrac{c_t^2}{1+n}\big)\Big], \tag{3.2}$$

where $c_t = c_t^1 + \frac{c_t^2}{1+n}$ is total consumption per capita in period t. Using the accumulation law $E : \mathbb{R}_+^2 \to \mathbb{R}_+$ for the pollution stock defined in (2.18), a *feasible allocation* may now formally be defined as follows.

Definition 1 (Feasible allocation).
Given the initial condition $(k_0, e_0) \in \mathbb{R}_{++} \times \mathbb{R}_+$, a feasible allocation *is a sequence $\big\{(k_t, e_t, y_t^b, c_t^1, c_t^2)\big\}_{t=0}^{\infty}$ that satisfies*

$$\begin{cases} k_{t+1} &= A(k_t, y_t^b, c_t^1, c_t^2) \\ e_{t+1} &= E(e_t, y_t^b) \end{cases} \tag{3.3}$$

with $(y_t^b, c_t^1, c_t^2) \in Q(k_t)$ for all times $t \geq 0$. The set of all *feasible allocations, given*

(k_0, e_0), *is*

$$\Pi(k_0,e_0) \coloneqq \Big\{ \big\{(k_t, e_t, y_t^b, c_t^1, c_t^2)\big\}_{t=0}^{\infty} \,\Big|\, \forall\, t \geq 0,\ (y_t^b, c_t^1, c_t^2) \in Q(k_t), \\ k_{t+1} = A(k_t, y_t^b, c_t^1, c_t^2),\ \text{and}\ \ e_{t+1} = E(e_t, y_t^b) \Big\}.$$

Notice that our definition of a feasible allocation excludes factor allocations that are not efficient. The reason is that a market economy as well as a benevolent social planner, as introduced in the next section, will only implement efficient factor allocations. In particular, if a production plan were inefficient, then the output in the non-polluting sector could always be raised and welfare be increased without lowering the output in the brown sector.

3.2 The social planner's problem

We first formalize the social planner's objective function. Taking a utilitarian measure, the welfare of the generation born in period $t \geq 0$ is $U(c_t^1, c_{t+1}^2, z_t, z_{t+1})$ as defined in (2.21) and the welfare of the initial old generation is $\beta[u(c_0^2) - \mu(z_0)]$. The planner's objective function, henceforth referred to as *social welfare function*, then takes the form

$$\mathcal{W}\big(\{(k_t, e_t, y_t^b, c_t^1, c_t^2)\}_{t=0}^{\infty}\big) \coloneqq \sum_{t=0}^{\infty} \gamma^t g(k_t, e_t, y_t^b, c_t^1, c_t^2), \tag{3.4}$$

where $0 < \gamma < 1$ is the social discount factor and

$$g(k_t, e_t, y_t^b, c_t^1, c_t^2) \coloneqq u(c_t^1) + \tfrac{\beta}{\gamma} u(c_t^2) - \big(1 + \tfrac{\beta}{\gamma}\big)\mu\big((1-\zeta)e_t + \epsilon y_t^b\big) \tag{3.5}$$

is the one-period return function. The social planner's task is to select the feasible allocation $\big\{(k_t, e_t, y_t^b, c_t^1, c_t^2)\big\}_{t=0}^{\infty} \in \Pi(k_0, e_0)$ with the highest possible level of social welfare. In each period $t \geq 0$, the planner chooses a production plan $(y_t^b, T(k_t, y_t^b))$ on the production-possibility frontier, a consumption plan (c_t^1, c_t^2), and an investment level k_{t+1}. Given the initial condition $(k_0, e_0) \in \mathbb{R}_{++} \times \mathbb{R}_+$, the planner's problem is

$$\max \Big\{ \mathcal{W}\big(\{(k_t, e_t, y_t^b, c_t^1, c_t^2)\}_{t=0}^{\infty}\big) \,\Big|\, \big\{(k_t, e_t, y_t^b, c_t^1, c_t^2)\big\}_{t=0}^{\infty} \in \Pi(k_0, e_0) \Big\}. \tag{3.6}$$

A solution $\big\{(k_t^*, e_t^*, y_t^{b*}, c_t^{1*}, c_t^{2*})\big\}_{t=0}^{\infty}$ to Problem (3.6) will be referred to as an *optimal allocation.*

To solve the planning problem using dynamic programming methods, a number of technical results found in De La Croix and Michel (2002) must be adapted. Proposition 1 ensures that Problem (3.6) is well posed.

Proposition 1 (Existence of the value function)**.**
Let Assumptions 1 and 2 be satisfied. Then for each $(k_0, e_0) \in \mathbb{R}_{++} \times \mathbb{R}_+$, *the value*

function

$$\mathcal{V}(k_0,e_0) \coloneqq \sup\Big\{\mathcal{W}\big(\{(k_t,e_t,y_t^b,c_t^1,c_t^2)\}_{t=0}^{\infty}\big) \;\Big|\; \{(k_t,e_t,y_t^b,c_t^1,c_t^2)\}_{t=0}^{\infty} \in \Pi(k_0,e_0)\Big\}$$

is well defined and finite.

All relevant properties of the value function $\mathcal{V}$ are summarised in Lemma 2.

Lemma 2 (Properties of the value function)**.**
Let Assumptions 1 and 2 be satisfied. Then for each $(k_0,e_0) \in \mathbb{R}_{++} \times \mathbb{R}_+$, the value function $\mathcal{V} : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}$ satisfies the Bellman equation

$$\mathcal{V}(k_0,e_0) = \sup\Big\{g(k_0,e_0,y^b,c^1,c^2) + \gamma\mathcal{V}\big(A(k_0,y^b,c^1,c^2),E(e_0,y^b)\big) \;\Big|\; (y^b,c^1,c^2) \in Q(k_0)\Big\}.$$

Moreover, $\mathcal{V}$ is concave, continuous, and differentiable.

Using the Bellman equation, the existence of a unique optimal allocation and its characterisation via first-order conditions are established in the following theorem.

Theorem 1 (Existence and uniqueness of the optimal allocation)**.**
Let Assumptions 1 and 2 be satisfied. Then the following holds true.

(i) For any given initial condition $(k_0,e_0) \in \mathbb{R}_{++} \times \mathbb{R}_+$, there exists a uniquely determined optimal allocation $\big\{(k_t^*,e_t^*,y_t^{b*},c_t^{1*},c_t^{2*})\big\}_{t=0}^{\infty} \in \Pi(k_0,e_0)$ that attains the welfare maximum,

$$\mathcal{V}(k_0,e_0) = \mathcal{W}\big(\{(k_t^*,e_t^*,y_t^{b*},c_t^{1*},c_t^{2*})\}_{t=0}^{\infty}\big),$$

where $(k_0^*,e_0^*) = (k_0,e_0)$. This optimal allocation is generated by continuous policy functions $y_*^b : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}_+$ and $c_*^1, c_*^2 : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}_{++}$ in the sense that for each $t \geq 0$,

$$\begin{aligned} k_{t+1}^* &= A(k_t^*,y_t^{b*},c_t^{1*},c_t^{2*}) > 0 \\ e_{t+1}^* &= E(e_t^*,y_t^{b*}) \\ (y_t^{b*},c_t^{1*},c_t^{2*}) &= \big(y_*^b(k_t^*,e_t^*),c_*^1(k_t^*,e_t^*),c_*^2(k_t^*,e_t^*)\big) \in Q(k_t^*). \end{aligned}$$

(ii) For each $t \geq 0$, the optimal allocation $\big\{(k_t^*,e_t^*,y_t^{b*},c_t^{1*},c_t^{2*})\big\}_{t=0}^{\infty}$ satisfies the first-order conditions

$$\frac{u'(c_t^{1*})}{\beta u'(c_t^{2*})} = \tfrac{1+n}{\gamma} \tag{3.7}$$

$$\frac{u'(c_t^{1*})}{\beta u'(c_{t+1}^{2*})} = \frac{\partial f}{\partial k}(k_{t+1}^*,y_{t+1}^{b*}) + \lambda_{t+1}^2 f_b'(k_{t+1}^*) \tag{3.8}$$

$$u'(c_t^{1*})\Big[\frac{\partial f}{\partial y^b}(k_t^*,y_t^{b*}) + \lambda_t^1 - \lambda_t^2\Big] = \epsilon(1+\tfrac{\beta}{\gamma})\sum_{j=t}^{\infty}\Big[\tfrac{\gamma(1-\zeta)}{1+n}\Big]^{j-t}\mu'\big((1+n)e_{j+1}^*\big) \tag{3.9}$$

together with the complementary slackness conditions

$$\lambda_t^1 y_t^{b*} = 0 \quad \text{and} \quad \lambda_t^2\big[f_b(k_t^*) - y_t^{b*}\big] = 0, \qquad \lambda_t^1, \lambda_t^2 \geq 0. \tag{3.10}$$

While closed-form solutions for the policy functions are generally unavailable, the first-order conditions (3.7) – (3.9) have a clear economic intuition. Equation (3.7) is the standard condition for OLG models that determines the optimal allocation of consumption between two coexisting generations. Equation (3.8) specifies the agent's marginal rate of intertemporal substitution. Thus, it determines the optimal allocation of consumption within an individual's lifetime.

Equation (3.9) will be of central importance for analysing the optimal taxation of emissions. It equates the marginal utility of producing an additional unit of brown output in period t with the sum of the discounted marginal damages incurred by all generations living from period t onward.[4] The factor $(1 + \frac{\beta}{\gamma})$ accounts for the fact that at any point in time, pollution affects the welfare of the young and the old generation. The Lagrange multipliers λ_t^1, λ_t^2 are needed because the optimal allocation may include boundary factor allocations. The complementary slackness conditions (3.10) imply that these are zero whenever both sectors are producing.

Two special cases in which the first-order condition (3.9) allows for a more tractable characterisation of the policy function y_*^b are presented in the following corollary.

Corollary 1.
For each $(k_t, e_t) \in \mathbb{R}_{++} \times \mathbb{R}_+$, the optimal policy $y_t^{b*} = y_*^b(k_t, e_t)$ satisfies the following properties.

(i) If $\zeta = 1$, then

$$y_t^{b*} = \begin{cases} 0 & \text{if} \quad \varrho(\Omega_g(k_t)) \geq 1 - \epsilon\big(1 + \frac{\beta}{\gamma}\big)\dfrac{\mu'(0)}{u'(c_*^1(k_t, e_t))} \\[2ex] f_b(k_t) & \text{if} \quad \varrho(\Omega_b(k_t)) \leq 1 - \epsilon\big(1 + \frac{\beta}{\gamma}\big)\dfrac{\mu'(\epsilon f_b(k_t))}{u'(c_*^1(k_t, e_t))} \\[2ex] \text{solves} & \varrho(\Omega(k_t, y_t^{b*})) = 1 - \epsilon\big(1 + \frac{\beta}{\gamma}\big)\dfrac{\mu'(\epsilon y_t^{b*})}{u'(c_*^1(k_t, e_t))} \quad \text{otherwise} \end{cases}.$$

(ii) If $\mu' \equiv 0$, then

$$y_t^{b*} = \underset{0 \leq y^b \leq f_b(k_t)}{\operatorname{argmax}}\ f(k_t, y^b) = \begin{cases} 0 & \text{if } \varrho(\Omega_g(k_t)) \geq 1 \\ f_b(k_t) & \text{if } \varrho(\Omega_b(k_t)) \leq 1 \\ \text{solves } \varrho(\Omega(k_t, y_t^{b*})) = 1 & \text{otherwise} \end{cases}.$$

[4]First-order conditions of the form (3.9) are often encountered in the literature on intergenerational pollution externalities, e.g., see Equation (22) in Jaimes (2023).

Corollary 1 (i) characterises the optimal production plan $(y_t^{b*}, T(k_t^*, y_t^{b*}))$ if the pollution stock decays fully between any two periods, so that only the instantaneous emissions $\epsilon\, y_t^{b*}$ affect the welfare of the two generations in period t. The optimal level of brown output y_t^{b*} then depends on how the marginal rate of transformation relates to the marginal rate of substitution between consumption and pollution. Corollary 1 (ii) states that in a model without externalities, the optimal production plan $(y_t^{b*}, T(k_t^*, y_t^{b*}))$ maximises the total output of the economy, given the capital-labour ratio k_t^*.

3.3 Modified golden-rule steady states

A *stationary feasible allocation* is a constant sequence $\{(\bar{k}, \bar{e}, \bar{y}^b, \bar{c}^1, \bar{c}^2)\}$ that is feasible in the sense of Definition 1. The second equation in (3.3) implies that the stationary pollution stock per capita $\bar{e}$ satisfies $\bar{e} = E(\bar{e}, \bar{y}^b)$. On the other hand, it follows from the first equation in (3.3) that for any $(\bar{k}, \bar{y}^b) \in \mathcal{D}$, stationary total consumption per capita $\bar{c} = \bar{c}^1 + \frac{\bar{c}^2}{1+n}$ is stipulated by the function

$$\bar{c} = \phi(\bar{k}, \bar{y}^b) \coloneqq f(\bar{k}, \bar{y}^b) - (1+n)\bar{k}. \tag{3.11}$$

By Theorem 1, the dynamics induced by the social planner's optimal policy is governed by the dynamical system

$$\begin{cases} k_{t+1} &= A\big(k_t, y_*^b(k_t, e_t), c_*^1(k_t, e_t), c_*^2(k_t, e_t)\big) \\ e_{t+1} &= E\big(e_t, y_*^b(k_t, e_t)\big) \end{cases}. \tag{3.12}$$

A steady state of the system (3.12), which will be referred to as a *modified golden-rule steady state*, is a stationary feasible allocation $\{(\bar{k}_\gamma, \bar{e}_\gamma, \bar{y}_\gamma^b, \bar{c}_\gamma^1, \bar{c}_\gamma^2)\}$ with $(\bar{y}_\gamma^b, \bar{c}_\gamma^1, \bar{c}_\gamma^2) \in Q(\bar{k}_\gamma)$ that satisfies

$$\begin{aligned} \bar{k}_\gamma &= A(\bar{k}_\gamma, \bar{y}_\gamma^b, \bar{c}_\gamma^1, \bar{c}_\gamma^2) \\ \bar{e}_\gamma &= E(\bar{e}_\gamma, \bar{y}_\gamma^b). \end{aligned} \tag{3.13}$$

Using the function ϕ defined in (3.11), the steady-state conditions in (3.13) take the more convenient form

$$\bar{c}_\gamma^1 + \tfrac{\bar{c}_\gamma^2}{1+n} = \phi(\bar{k}_\gamma, \bar{y}_\gamma^b) \tag{3.14}$$

$$\bar{e}_\gamma = \tfrac{\epsilon}{n+\zeta}\, y_\gamma^b. \tag{3.15}$$

The social planner's first-order condition (3.7) implies that the marginal rate of intertemporal substitution in a steady state fulfils

$$\frac{u'(\bar{c}_\gamma^1)}{\beta u'(\bar{c}_\gamma^2)} = \tfrac{1+n}{\gamma}. \tag{3.16}$$

Thus, given any pair $(\bar{k}_\gamma, \bar{y}^b_\gamma) \in \mathcal{D}$ with $\phi(\bar{k}_\gamma, \bar{y}^b_\gamma) > 0$, the steady-state consumption plan

$$\bar{c}^1_\gamma = \bar{c}_1(\bar{k}_\gamma, \bar{y}^b_\gamma) \quad \text{and} \quad \bar{c}^2_\gamma = \bar{c}_2(\bar{k}_\gamma, \bar{y}^b_\gamma)$$

is uniquely determined by the *budget constraint* (3.14) and the first-order condition (3.16). As a consequence, the modified golden-rule steady state is determined by the pair $(\bar{k}_\gamma, \bar{y}^b_\gamma)$. The central question now is how $(\bar{k}_\gamma, \bar{y}^b_\gamma)$ must be chosen such that the social welfare is maximal. Using (3.9), we may define the function $\psi : \mathcal{D} \to \mathbb{R}_+$ by setting

$$\psi(k, y^b) := \frac{\epsilon\big(1 + \frac{\beta}{\gamma}\big)}{1 - \frac{\gamma(1-\zeta)}{1+n}} \frac{\mu'\big(\frac{(1+n)\epsilon}{n+\zeta}\, y^b\big)}{u'\big(\bar{c}_1(k, y^b)\big)}, \tag{3.17}$$

which contains a marginal rate of substitution that describes the trade-off between the steady-state consumption of the two generations and the steady-state pollution level. For each sector $j = g, b$, we introduce the capital-labour ratio $\bar{k}^j > 0$, defined by

$$f_j'(\bar{k}^j) = \tfrac{1+n}{\gamma} - 1 + \delta,$$

and the corresponding wage-rental ratio $\bar{\omega}^j = \Omega_j(\bar{k}^j)$. We are now in a position to state one of this article's main results.

Theorem 2 (Modified golden-rule steady states).
Under the hypotheses of Assumptions 1 and 2, there exists at least one modified golden-rule steady state $(\bar{k}_\gamma, \bar{y}^b_\gamma)$. These steady states are characterised as follows.

(i) If $1 - \psi(\kappa_g(\bar{\omega}^g), 0) \le \varrho(\bar{\omega}^g)$, then $(\bar{k}_\gamma, \bar{y}^b_\gamma) = (\bar{k}^g, 0)$ is a steady state.

(ii) If either

$$1 - \psi(\kappa_g(\bar{\omega}^g), 0) < \varrho(\bar{\omega}^g) < 1 - \psi\big(\kappa_b(\bar{\omega}^g), f_b(\kappa_b(\bar{\omega}^g))\big) \tag{3.18}$$

or

$$1 - \psi(\kappa_g(\bar{\omega}^g), 0) > \varrho(\bar{\omega}^g) > 1 - \psi\big(\kappa_b(\bar{\omega}^g), f_b(\kappa_b(\bar{\omega}^g))\big), \tag{3.19}$$

then there exists $\bar{k}_\gamma \in \big(\min\{\kappa_g(\bar{\omega}^g), \kappa_b(\bar{\omega}^g)\}, \max\{\kappa_g(\bar{\omega}^g), \kappa_b(\bar{\omega}^g)\}\big)$ determined by

$$\varrho(\bar{\omega}^g) = 1 - \psi\big(\bar{k}_\gamma, \mathsf{y}_b(\bar{k}_\gamma, \bar{\omega}^g)\big) \tag{3.20}$$

and $\bar{y}^b_\gamma = \mathsf{y}_b(\bar{k}_\gamma, \bar{\omega}^g)$ such that $(\bar{k}_\gamma, \bar{y}^b_\gamma)$ is a steady state.

(iii) If $\varrho(\bar{\omega}^g) \le 1 - \psi\big(\kappa_b(\bar{\omega}^g), f_b(\kappa_b(\bar{\omega}^g))\big)$, then there exists $\bar{k}_\gamma \in \big[\kappa_b(\bar{\omega}^g), \bar{k}^b\big)$ determined by

$$f_b'(\bar{k}_\gamma)\big[1 - \psi(\bar{k}_\gamma, f_b(\bar{k}_\gamma))\big] = \tfrac{1+n}{\gamma} - 1 + \delta$$

and $\bar{y}^b_\gamma = f_b(\bar{k}_\gamma)$ such that $(\bar{k}_\gamma, \bar{y}^b_\gamma)$ is a steady state.

The three distinct types of steady states in Theorem 2 are referred to as green, mixed, and brown, respectively. For the intuition of theorem, the following corollary is insightful.

Corollary 2.
If $\mu' \equiv 0$, then there exists a modified golden-rule steady state, determined by

$$(\bar{k}_\gamma, \bar{y}^b_\gamma) = \begin{cases} (\bar{k}^g, 0) & \text{if } \varrho(\bar{\omega}^g) \geq 1 \\ \big(\bar{k}^b, f_b(\bar{k}^b)\big) & \text{if } \varrho(\bar{\omega}^g) \leq 1 \end{cases}, \tag{3.21}$$

where $f_g'(\bar{k}^g) = f_b'(\bar{k}^b) = \frac{1+n}{\gamma} - 1 + \delta$. If, in addition, $\gamma = 1$, then ϕ attains its maximum in $(\bar{k}_\gamma, \bar{y}^b_\gamma)$ and the consumption levels are $\phi(\bar{k}^g, 0) = w_g(\bar{k}^g)$ and $\phi(\bar{k}^b, f_b(\bar{k}^b)) = w_b(\bar{k}^b)$, respectively.

Corollary 2 reveals that if pollution plays no role, then the modified golden-rule steady state permits only the more productive sector to produce the good.[5] In the presence of pollution, however, the type of the steady state does not only depend the sectors' relative productivity, but also on the emission intensity ϵ, the compound discount factor $\frac{\gamma(1-\zeta)}{1+n}$, and agents' preferences, especially their valuation of the externality. The resulting trade-off is described by the relationship between the marginal rate of transformation ϱ and the marginal rate of substitution ψ. In the long-run, complete decarbonisation is optimal if and only if $1 - \psi(\bar{k}^g, 0) \leq \varrho(\bar{\omega}^g)$. Since $\psi \geq 0$, a unique green steady state exists if $\varrho(\bar{\omega}^g) \geq 1$, in which case the green technology is at least as productive as the polluting technology. If $\varrho(\bar{\omega}^g) < 1 - \psi(\bar{k}^g, 0)$, then it is optimal to let brown firms produce, as the consumption benefits outweigh the pollution damages. The exact extent, however, depends on the severity of the externality.

The intuition of Theorem 2 is linked to the well-known *Rybczynski-Theorem.* In Ritschel and Wenzelburger (2024), we show that all efficient production plans with the same marginal rate of transformation $\varrho(\bar{\omega}^g)$ are located on a straight line

$$y^g = f_g(\kappa_g(\bar{\omega}^g)) - \frac{f_g(\kappa_g(\bar{\omega}^g))}{f_b(\kappa_b(\bar{\omega}^g))}\, y^b, \quad y^b \in \big[0, f_b(\kappa_b(\bar{\omega}^g))\big],$$

in the (y^b, y^g)-plane. With this line, we may associate a family of production-possibility frontiers corresponding to the capital-labour ratios

$$\bar{k} \in \big(\min\{\kappa_g(\bar{\omega}^g), \kappa_b(\bar{\omega}^g)\}, \max\{\kappa_g(\bar{\omega}^g), \kappa_b(\bar{\omega}^g)\}\big). \tag{3.22}$$

If the interval (3.22) contains a capital-labour ratio $\bar{k}_\gamma$ such that

$$\varrho(\bar{\omega}^g) = 1 - \psi\big(\bar{k}_\gamma, \mathsf{y}_b(\bar{k}_\gamma, \bar{\omega}^g)\big),$$

then this capital-labour ratio balances the trade-off between consumption and pollution and, thus, defines the mixed steady state. If such a capital-labour ratio does not exist, then a mixed steady state does not exist. Instead, a green or a brown boundary steady

[5]Observe that $\varrho(\bar{\omega}^g) \geq 1 \iff w_g(\bar{k}^g) \geq w_b(\bar{k}^b)$, so that the stationary wage income determines which sector is active. The case $\gamma = 1$ corresponds to the *modified golden rule*, i.e., the question of which stationary feasible allocation has the highest level of welfare.

state is obtained.

Importantly, observe that mixed steady states may exist either due to (3.18) or due to (3.19). In the former case, it follows directly from Theorem 2 that a green, a brown, and a mixed steady state *coexist.*[6] Which of these steady states is attained depends on their stability properties and the initial conditions of the economy.

The following example adopts standard preferences and technologies from the literature.

Example 1 (Logarithmic utility and Cobb-Douglas technology)**.**
Consider the logarithmic utility function $u(c) = \ln(c)$*, the damage function* $\mu(z) = dz^\sigma$*, where* $d > 0$ *and* $\sigma \geq 1$*, and the Cobb-Douglas production functions*

$$f_g(k) = A_g k^{\alpha_g} \quad \textit{and} \quad f_b(k) = A_b k^{\alpha_b},$$

where $A_g, A_b > 0$ *scale total factor productivity and* $0 < \alpha_g, \alpha_b < 1$ *determine the income distribution. In this case, the function* ϱ *becomes*

$$\varrho(\omega) = \left(\frac{\alpha_b^{\alpha_b}(1-\alpha_b)^{1-\alpha_b} A_b}{\alpha_g^{\alpha_g}(1-\alpha_g)^{1-\alpha_g} A_g} \right) \omega^{\alpha_b - \alpha_g}. \tag{3.23}$$

It follows from (3.14) *and* (3.16) *that the steady-state consumption plan satisfies*

$$\bar{c}^1(k, y^b) = \frac{\phi(k, y^b)}{1 + \frac{\beta}{\gamma}} \quad \textit{and} \quad \bar{c}^2(k, y^b) = (1+n)\frac{\phi(k, y^b)}{1 + \frac{\gamma}{\beta}},$$

so that the function ψ *takes the form*

$$\psi(k, y^b) = \frac{\epsilon d \sigma \left[\frac{(1+n)\epsilon}{n+\zeta} y^b \right]^{\sigma - 1}}{1 - \frac{\gamma(1-\zeta)}{1+n}} \phi(k, y^b).$$

A green steady state is characterised by

$$\bar{k}^g = \left(\frac{A_g \alpha_g}{\frac{1+n}{\gamma} - 1 + \delta} \right)^{\frac{1}{1-\alpha_g}} \quad \textit{and} \quad \bar{\omega}^g = \Omega_g(\bar{k}^g) = \tfrac{1-\alpha_g}{\alpha_g} \bar{k}^g.$$

The capital-labour ratio in a brown steady state is determined by

$$A_b \alpha_b (\bar{k}_\gamma)^{\alpha_b - 1} \left[1 - \frac{\epsilon d \sigma \left[\frac{(1+n)\epsilon}{n+\zeta} A_b (\bar{k}_\gamma)^{\alpha_b} \right]^{\sigma-1}}{1 - \frac{\gamma(1-\zeta)}{1+n}} \left[A_b (\bar{k}_\gamma)^{\alpha_b} - (n+\delta)\bar{k}_\gamma \right] \right] = \tfrac{1+n}{\gamma} - 1 + \delta.$$

For parameter set A, see Table 1, we have $\bar{\omega}^g = 0.456$*. In this case, all three types of steady states coexist. The capital-labour ratio in the green steady is* 1.617*, in the brown steady state* 0.489*, and in the mixed steady state* 1.419*. For parameter set B, we have* $\bar{\omega}^g = 1.103$*. There exists a uniquely determined mixed steady state, characterised by the capital-labour ratio* 1.035*.*

[6] If $\mu'(0) = 0$, then $\psi(\bar{k}^g, 0) = 0$ such that different types of steady states cannot coexist.

Parameter	Set A	Set B
ϵ	1	1
ζ	0.9	0.9
σ	1	1
d	0.04	0.04
γ	0.5	0.5
β	1	1
n	0	0
δ	1	1
A_b	3	5
A_g	2.85	4.2
α_b	0.5	0.5
α_g	0.78	0.45

Table 1: Parameter sets for Example 1.

4 Implementation of Optimal Allocations

This section examines how fiscal policy can implement the social planner's optimal allocation in a competitive market economy.

4.1 Temporary equilibria

In a temporary equilibrium, all markets clear simultaneously and, given expectations, individual decisions are optimal. Since the sectors' outputs are perfect substitutes, Walras' law implies that the goods market must clear if the factor allocation is efficient. Recall that capital and labour are perfectly mobile between the sectors and are paid their marginal products (2.4). Hence, given any capital-labour ratio k_t, the equilibrium wage-rental ratio ω_t corresponding to an equilibrium production plan $(y_t^b, T(k_t, y_t^b))$ with $0 < y_t^b < f_b(k_t)$, i.e. with positive output in both sectors, must satisfy the first-order condition

$$\varrho(\omega_t) \stackrel{!}{=} \frac{p_t^b}{p_t^g}, \tag{4.1}$$

where p_t^j, $j = g, b$, are the sector-specific output prices in (2.4).

Suppose now that the government levies a tax rate η_t per unit of emissions in period t, which is stipulated by an *emissions-tax-policy rule* $\eta : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}$ such that $\eta_t = \eta(k_t, e_t)$.[7] Normalising the price of the *'green'* output, we may set $p_t^g = 1$ and $p_t^b = 1 - \epsilon\eta_t$ so that the output-price ratio becomes $\frac{p_t^b}{p_t^g} = 1 - \epsilon\eta_t$. We can now characterise the wage-rental ratio in a temporary equilibrium.

Proposition 2 (Existence and uniqueness of temporary equilibria).
Let Assumptions 1 and 2 be satisfied. Then for any state $(k_t, e_t) \in \mathbb{R}_{++} \times \mathbb{R}_+$ *with*

[7]Our approach also allows for the case $\eta_t < 0$, in which the brown sector is subsidised.

emissions tax rate $\eta_t = \eta(k_t, e_t) \in \mathbb{R}$*, the equilibrium wage-rental ratio is*

$$\omega_t = \Omega_{\text{eq}}(k_t, \eta_t) := \begin{cases} \Omega_g(k_t) & \textit{if } \varrho(\Omega_g(k_t)) \geq 1 - \epsilon\eta_t \\ \Omega_b(k_t) & \textit{if } \varrho(\Omega_b(k_t)) \leq 1 - \epsilon\eta_t \\ \varrho^{-1}(1 - \epsilon\eta_t) & \textit{otherwise} \end{cases}.$$

Setting $\omega_t = \Omega_{\text{eq}}(k_t, \eta(k_t, e_t))$*, the equilibrium wage rate is*

$$w_t = w(k_t, e_t) := \max\Big\{ w_g(\kappa_g(\omega_t)), \big[1 - \epsilon\eta(k_t, e_t)\big] w_b(\kappa_b(\omega_t)) \Big\}, \tag{4.2}$$

the equilibrium gross return on capital is

$$R_t = R(k_t, e_t) := 1 - \delta + \max\Big\{ f_g'(\kappa_g(\omega_t)), \big[1 - \epsilon\eta(k_t, e_t)\big] f_b'(\kappa_b(\omega_t)) \Big\}, \tag{4.3}$$

and the equilibrium production plan $(y_t^b, T(k_t, y_t^b))$ *is determined by*

$$y_t^b = \mathsf{y}_{\text{eq}}^b(k_t, e_t) := \begin{cases} \mathsf{y}_b(k_t, \omega_t) & \textit{if } \Omega_b(k_t) \neq \Omega_g(k_t) \\ f_b(k_t) & \textit{if } \varrho(\Omega_b(k_t)) = \varrho(\Omega_g(k_t)) < 1 - \epsilon\eta(k_t, e_t) \\ 0 & \textit{if } \varrho(\Omega_b(k_t)) = \varrho(\Omega_g(k_t)) \geq 1 - \epsilon\eta(k_t, e_t) \end{cases}. \tag{4.4}$$

By Proposition 2, the emissions tax rate η_t stipulates a uniquely determined equilibrium wage-rental ratio ω_t and thus the production plan $(y_t^b, T(k_t, y_t^b))$. The boundary production plans in which only one sector is producing obtain for sufficiently high and sufficiently low tax rates.

4.2 Fiscal policy

Proposition 2 implies that the sum of factor incomes in a temporary equilibrium is

$$w(k_t, e_t) + R(k_t, e_t)k_t = f(k_t, y_t^b) - \eta_t \epsilon y_t^b, \tag{4.5}$$

where $\eta_t \epsilon y_t^b$ are the government's emissions tax receipts in period t. These reduce the incomes of both generations. Assume for simplicity that the government does not hold or issue bonds and suppose that in addition to the emissions tax, it levies a proportional tax $\tau_t = \tau(k_t, e_t)$ on the wage income of young agents, which is stipulated by an *income-tax-policy rule* $\tau : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}$. If the government pays each old agent a lump-sum transfer $d_t \in \mathbb{R}$, then its budget constraint takes the form

$$\tfrac{d_t}{1+n} = \eta_t \epsilon y_t^b + \tau_t w_t. \tag{4.6}$$

Given a pair of tax-policy rules η, τ, the disposable income of a young agent in period t if the state of the economy is (k_t, e_t) is

$$w_t^d = w^d(k_t, e_t) := \big[1 - \tau(k_t, e_t)\big] w(k_t, e_t),$$

and the disposable income of an old agent amounts to

$$\pi_t = \pi(k_t, e_t) \coloneqq (1+n)R(k_t, e_t)k_t + d(k_t, e_t),$$

where the lump-sum transfer to an old agent is

$$d_t = d(k_t, e_t) \coloneqq (1+n)\big[\eta(k_t, e_t)\epsilon\, \mathsf{y}^b_{\mathsf{eq}}(k_t, e_t) + \tau(k_t, e_t)w(k_t, e_t)\big].$$

A fiscal policy is feasible if the disposable income of both generations is non-negative, i.e. $w_t^d \geq 0$ and $\pi_t \geq 0$. The former holds whenever $\tau_t \leq 1$; the latter holds if and only if

$$\tau_t \geq \underline{\tau}(k_t, \eta(k_t, e_t)) \coloneqq -\frac{R(k_t, e_t)k_t + \epsilon\, \eta(k_t, e_t)\mathsf{y}^b_{\mathsf{eq}}(k_t, e_t)}{w(k_t, e_t)}.$$

More formally, a *feasible fiscal policy* may be defined as follows.

Definition 2 (Feasible fiscal policy)**.**
A feasible fiscal policy *is a pair of tax-policy rules* $\eta, \tau : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}$ *that satisfies* $\underline{\tau}(k, \eta(k, e)) \leq \tau(k, e) \leq 1$ *for all* $(k, e) \in \mathbb{R}_{++} \times \mathbb{R}_+$.

We will assume for the remainder of this article that the government's fiscal policy is feasible.[8] Since Definition 2 allows for $\tau_t < 0$ and $d_t < 0$, the government may compensate the reduction in factor incomes by redistributing the emissions tax receipts in full to young and old agents. In particular, if $\tau_t = \underline{\tau}(k_t, \eta_t)$, then $\pi_t = 0$ and $w_t^d = f(k_t, y_t^b) > w_t$. On the other hand, if $\tau_t = 1$, then $\pi_t = (1+n)f(k_t, y_t^b)$ and $w_t^d = 0$.

4.3 Capital accumulation

The savings decision problem of a typical young agent is the following. In each period t, the agent forms an expectation $R_t^e > 0$ with respect to the gross return on savings R_{t+1} realised in $t+1$, and an expectation $d_t^e \in \mathbb{R}$ with respect to the transfer payment d_{t+1}. Given the disposable income w_t^d and the expectations (R_t^e, d_t^e), the savings decision problem takes the form[9]

$$\begin{aligned} &\max_s \; u(w_t^d - s) - \mu(z_t) + \beta\big[u(R_t^e s + d_t^e) - \mu(z_{t+1})\big] \\ &\text{s.t.} \quad \max\big\{\tfrac{-d_t^e}{R_t^e}, 0\big\} \leq s \leq w_t^d. \end{aligned} \tag{4.7}$$

The two constraints in (4.7) ensure that savings, youthful consumption, and anticipated old-age consumption are non-negative. Given that the agent's anticipated lifetime income is non-negative, $w_t^d + \frac{d_t^e}{R_t^e} \geq 0$, then Problem (4.7) admits a uniquely determined solution $s_t = s(w_t^d, d_t^e, R_t^e)$.[10] An interior solution $s(w_t^d, d_t^e, R_t^e) > 0$ obtains from the first-order

[8] Since $\underline{\tau}(k_t, \eta_t) \leq 0$ whenever $\eta_t \geq 0$, any fiscal policy defined by tax-policy rules of the form $\eta : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\tau : \mathbb{R}_{++} \times \mathbb{R}_+ \to [0, 1]$ is feasible.

[9] Since each individual agent has mass zero, the agent does not take into account how his savings behaviour affects the pollution indices z_t and z_{t+1}.

[10] In the extreme case $w_t^d + \frac{d_t^e}{R_t^e} = 0$, savings are $s_t = w_t^d = -\frac{d_t^e}{R_t^e}$.

condition

$$\frac{u'(w_t^d - s_t)}{\beta u'(R_t^e s_t + d_t^e)} = R_t^e. \tag{4.8}$$

The capital-labour ratio of the subsequent period $t+1$ is then given by

$$k_{t+1} = \tfrac{1}{1+n} s(w_t^d, d_t^e, R_t^e).$$

Note that k_{t+1} is only well defined for forecasts (R_t^e, d_t^e) satisfying $w_t^d + \frac{d_t^e}{R_t^e} \geq 0$.

4.4 Perfect-foresight dynamics

The expectations formation is considered next. In period $t+1$, the realised gross return on savings is $R_{t+1} = R(k_{t+1}, e_{t+1})$, and the realised transfer payment to an old agent is $d_{t+1} = d(k_{t+1}, e_{t+1})$, where

$$e_{t+1} = E_{\text{eq}}(k_t, e_t) \coloneqq E(e_t, \mathsf{y}^b_{\text{eq}}(k_t, e_t))$$

governs the accumulation of emissions. If k_t^e is the forecast for k_{t+1}, the forecasts (R_t^e, d_t^e) that are consistent with k_t^e obtain by setting

$$R_t^e = R(k_t^e, E_{\text{eq}}(k_t, e_t)) \quad \text{and} \quad d_t^e = d(k_t^e, E_{\text{eq}}(k_t, e_t)).$$

Observe that these forecasts are correct whenever k_t^e correctly predicts k_{t+1}. Our next lemma ensures that the decision problem of a young agent is well posed.

Lemma 3 (Anticipated lifetime income)**.**
Let Assumptions 1 and 2 be satisfied and the fiscal policy be feasible. Then for any state $(k_t, e_t) \in \mathbb{R}_{++} \times \mathbb{R}_+$ *and each forecast* $0 \leq k_t^e \leq \frac{1}{1+n} w^d(k_t, e_t)$*, a young agent's anticipated lifetime income is positive, that is,*

$$w^d(k_t, e_t) + \frac{d(k_t^e, E_{\text{eq}}(k_t, e_t))}{R(k_t^e, E_{\text{eq}}(k_t, e_t))} \geq 0. \tag{4.9}$$

Since the agent's anticipated lifetime income is positive under the assumptions of Lemma 3, the savings function is well defined. Observe that a forecast $k_t^e > \frac{1}{1+n} w_t^d$ can never be a self-fulfilling prophecy because savings cannot exceed the disposable income.

Given (k_t, e_t), the forecast k_t^e is correct, i.e. $k_t^e = k_{t+1}$, if it solves

$$k_t^e = \tfrac{1}{1+n} s\big(w^d(k_t, e_t), d(k_t^e, E_{\text{eq}}(k_t, e_t)), R(k_t^e, E_{\text{eq}}(k_t, e_t))\big).$$

A *perfect forecasting rule* in the sense of Böhm and Wenzelburger (1999) may now be defined as follows.

Definition 3 (Perfect forecasting rule)**.**

A forecasting rule $G : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}_+$ *that satisfies*

$$G(k,e) = \tfrac{1}{1+n} s\big(w^d(k,e), d(G(k,e),e_1), R(G(k,e),e_1)\big), \quad \textit{where} \quad e_1 = E_{\mathrm{eq}}(k,e),$$

for all $(k,e) \in \mathbb{R}_{++} \times \mathbb{R}_+$ *is called a* perfect forecasting rule.

The existence of a perfect forecasting rule is established next.[11]

Proposition 3 (Existence of perfect forecasting rules)**.**
Under the hypotheses of Assumptions 1 and 2, there exists a perfect forecasting rule G *in the sense of Definition 3. If, in addition,* $R^e \mapsto s(w^d, d^e, R^e)$ *and* $k \mapsto d(k,e)$ *are non-decreasing, then* G *is uniquely determined.*

Given the initial condition $(k_0, e_0) \in \mathbb{R}_{++} \times \mathbb{R}_+$, all growth paths $\{(k_t, e_t)\}_{t=0}^{\infty}$ under perfect foresight are generated recursively by the perfect forecasting rule G together with the map E_{eq}, so that

$$\begin{cases} k_{t+1} &= G(k_t, e_t) \\ e_{t+1} &= E_{\mathrm{eq}}(k_t, e_t) \end{cases} . \tag{4.10}$$

With each growth path $\{(k_t, e_t)\}_{t=0}^{\infty}$ generated by the dynamical system (4.10), we may associate a *perfect-foresight allocation*, which is the feasible allocation $\big\{(k_t, e_t, y_t^b, c_t^1, c_t^2)\big\} \in \Pi(k_0, e_0)$ such that for each $t \geq 0$,

$$\begin{aligned} k_{t+1} &= G(k_t, e_t) \\ e_{t+1} &= E_{\mathrm{eq}}(k_t, e_t) \\ y_t^b &= \mathsf{y}_{\mathrm{eq}}^b(k_t, e_t) \\ c_t^1 &= w^d(k_t, e_t) - (1+n)G(k_t, e_t) \\ c_t^2 &= \pi(k_t, e_t). \end{aligned}$$

Given a pair of tax-policy rules η, τ, a *perfect-foresight steady state* $(k_\star, e_\star)$ of the dynamical system (4.10) is determined by

$$k_\star = \tfrac{1}{1+n} s\big(w^d(k_\star, e_\star), d(k_\star, e_\star), R(k_\star, e_\star)\big) \tag{4.11}$$

$$e_\star = E_{\mathrm{eq}}(k_\star, e_\star). \tag{4.12}$$

4.5 Optimal fiscal policies

Having established the perfect-foresight dynamics, we are now in a position to present another main result of this article.

Theorem 3 (Implementation of the optimal allocation)**.**
Let Assumptions 1 and 2 be satisfied. Then for any given initial condition $(k_0, e_0) \in$

[11] As is well known, $R^e \mapsto s(w^d, d^e, R^e)$ is non-decreasing if youthful and old-age consumption are weak gross substitutes.

$\mathbb{R}_{++} \times \mathbb{R}_+$ and corresponding optimal allocation $\big\{(k_t^*, e_t^*, y_t^{b*}, c_t^{1*}, c_t^{2*})\big\}_{t=0}^{\infty} \in \Pi(k_0, e_0)$, there exists a uniquely determined series of optimal tax rates $\{(\eta_t^*, \tau_t^*)\}_{t=0}^{\infty}$ such that $\big\{(k_t^*, e_t^*, y_t^{b*}, c_t^{1*}, c_t^{2*})\big\}_{t=0}^{\infty}$ is a perfect-foresight allocation. The series $\{(\eta_t^*, \tau_t^*)\}_{t=0}^{\infty}$ is generated by a feasible fiscal policy $\eta_*, \tau_* : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}$ such that for each $t \geq 0$, $\eta_t^* = \eta_*(k_t^*, e_t^*)$ and $\tau_t^* = \tau_*(k_t^*, e_t^*)$. The optimal emissions tax rate satisfies

$$\eta_t^* = \frac{1 + \frac{\beta}{\gamma}}{u'(c_t^{1*})} \sum_{j=t}^{\infty} \left[\tfrac{\gamma(1-\zeta)}{1+n}\right]^{j-t} \mu'(z_j^*). \tag{4.13}$$

Theorem 3 establishes the existence of a feasible fiscal policy in the sense of Definition 2 that decentralises the optimal allocation in a market economy with perfect foresight. This fiscal policy has the following properties.

The optimal emissions tax rate η_t^* implements the welfare-maximising production plan $(y_t^{b*}, T(k_t^*, y_t^{b*}))$ by implementing the optimal wage-rental ratio $\omega_t^* = \Omega(k_t^*, y_t^{b*})$, where the function Ω is given in Lemma 1. It follows that the optimal emissions-tax-policy rule $\eta_* : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}$ is implicitly defined by

$$\Omega_{\text{eq}}(k, \eta_*(k, e)) = \Omega(k, y_*^b(k, e)). \tag{4.14}$$

Equation (4.13) states that in each period $t \geq 0$, η_t^* is equal to the sum of the discounted marginal damages incurred by all future generations.

Given the optimal emissions tax rate η_t^*, the optimal income tax rate τ_t^* induces intergenerational transfers such that under perfect foresight, a young agent decides at his own discretion to save the amount of funds required to attain the capital-labour ratio k_{t+1}^*. Thus, given the functions η_*, G, and E_{eq}, the optimal income-tax-policy rule $\tau_* : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}$ is implicitly defined by

$$\begin{aligned} &A\big(k, c_*^1(k, e), c_*^2(k, e), y_*^b(k, e)\big) \\ &\quad = \tfrac{1}{1+n} s\Big([1 - \tau_*(k, e)]w(k, e), d\big(G(k, e), E_{\text{eq}}(k, e)\big), R\big(G(k, e), E_{\text{eq}}(k, e)\big)\Big). \end{aligned}$$

A direct implication of Theorem 3 is that the government can implement a modified golden-rule steady state $\big\{(\bar{k}_\gamma, \bar{e}_\gamma, \bar{y}_\gamma^b, \bar{c}_\gamma^1, \bar{c}_\gamma^2)\big\}$ as a perfect-foresight steady state. This result is formalised in the following corollary.

Corollary 3.
Let $(\bar{k}_\gamma, \bar{e}_\gamma)$ with $\bar{e}_\gamma = \frac{\epsilon}{n+\zeta} \bar{y}_\gamma^b$ be a modified golden-rule steady state. Then the emissions tax rate $\bar{\eta}_\gamma = \eta_*(\bar{k}_\gamma, \bar{e}_\gamma)$ is

$$\bar{\eta}_\gamma = \frac{\psi(\bar{k}_\gamma, \bar{y}_\gamma^b)}{\epsilon}, \tag{4.15}$$

and the income tax rate $\bar{\tau}_\gamma = \tau_*(\bar{k}_\gamma, \bar{e}_\gamma) \in \big(\underline{\tau}(\bar{k}_\gamma, \bar{\eta}_\gamma), 1\big)$ is uniquely determined by

$$\bar{k}_\gamma = \tfrac{1}{1+n} s\big((1 - \bar{\tau}_\gamma)\bar{w}_\gamma, \bar{d}_\gamma, \bar{R}_\gamma\big), \tag{4.16}$$

where

$$\bar{w}_\gamma = w(\bar{k}_\gamma, \bar{e}_\gamma), \quad \bar{R}_\gamma = R(\bar{k}_\gamma, \bar{e}_\gamma) = \tfrac{1+n}{\gamma}, \quad \textit{and} \quad \bar{d}_\gamma = (1+n)\big[\bar{\eta}_\gamma \epsilon \bar{y}^b_\gamma + \bar{\tau}_\gamma \bar{w}_\gamma\big].$$

In the proof of Corollary 3, we show that the government can not only implement the modified golden-rule steady state $(\bar{k}_\gamma, \bar{e}_\gamma)$, but that it can achieve *any* desired steady state $(\bar{k}, \bar{e}) \in \mathbb{R}_{++} \times \mathbb{R}_+$ with $(1+n)\bar{k} < f(\bar{k}, \bar{y}^b)$, $\bar{e} = \frac{\epsilon}{n+\zeta}\bar{y}^b$, and $0 \le \bar{y}^b \le f_b(\bar{k})$. However, the implementation of certain (non-welfare-maximising) steady states may necessitate subsidies for the polluting sector.

Equation (4.15) reveals that the optimal steady-state emissions tax rate $\bar{\eta}_\gamma$ takes into account the trade-off between stationary consumption and pollution, which is captured by the marginal rate of substitution ψ. The optimal steady-state income tax rate $\bar{\tau}_\gamma$ implements transfers such that savings are $(1+n)\bar{k}_\gamma$.

Example 2 (Logarithmic utility and Cobb-Douglas technology)**.**
Following on from Example 1, consider a quadratic damage function, i.e. $\sigma = 2$*. Let* $\delta = 1$ *so that capital depreciates fully. Since* $\psi(\bar{k}^g, 0) = 0$*, the modified golden-rule steady state is green and determined by* $(\bar{k}_\gamma, \bar{y}^b_\gamma) = (\bar{k}^g, 0)$ *if* $\varrho(\bar{\omega}^g) \ge 1$*. The corresponding steady-state tax rates are*

$$\bar{\eta}_\gamma = \frac{\psi(\bar{k}^g, 0)}{\epsilon} = 0 \quad \textit{and} \quad \bar{\tau}_\gamma = \frac{1 - \gamma \frac{\alpha_g}{1-\alpha_g} \frac{1+\beta}{\beta}}{1 + \frac{\gamma}{\beta}}. \tag{4.17}$$

Observe that

$$\underline{\tau}(\bar{k}_\gamma, \bar{\eta}_\gamma) = \underline{\tau}(\bar{k}^g, 0) = \tfrac{\alpha_g}{\alpha_g - 1} < \bar{\tau}_\gamma < 1,$$

showing that the fiscal policy (4.17) *is feasible. Wage income is taxed if*

$$\bar{\tau}_\gamma \ge 0 \quad \Longleftrightarrow \quad \gamma \le \tfrac{\beta}{1+\beta} \tfrac{1-\alpha_g}{\alpha_g} \tag{4.18}$$

and subsidised otherwise. Condition (4.18) *may hold either due to a high savings propensity or due to labour-intensive production in the green sector.*

A surprising result obtains for *mixed* modified golden-rule steady states. Since, by (3.20), the marginal rate of transformation satisfies $1 - \varrho(\bar{\omega}^g) = \psi(\bar{k}_\gamma, \bar{y}^b_\gamma)$, it follows from (4.15) that

$$\bar{\eta}_\gamma = \frac{1 - \varrho(\bar{\omega}^g)}{\epsilon}. \tag{4.19}$$

Thus, $\bar{\eta}_\gamma$ depends solely on the technologies and the brown sector's emissions intensity ϵ. In particular, it is *decreasing* in ϵ.

In the case of a linear damage function $\mu(z) = dz$ with slope $d > 0$, we may characterise the sequence $\{\eta^*_t\}_{t=0}^\infty$ near a mixed modified golden-rule steady state by means of a simple difference equation.

Proposition 4 (Constant marginal damage)**.**
Let Assumptions 1 and 2 be satisfied and assume that the marginal damage of pollution

is a constant $d > 0$. Then, in a neighbourhood of a mixed modified golden-rule steady state, the series $\{\eta_t^*\}_{t=0}^{\infty}$ satisfies the difference equation

$$\frac{\eta_{t+1}^*}{\eta_t^*} = \tfrac{\gamma}{1+n}\Big[1 - \delta + f_g'\big(\kappa_g(\varrho^{-1}(1 - \epsilon\eta_{t+1}^*))\big)\Big], \tag{4.20}$$

where (4.19) *is the unique steady-state emissions tax rate.*

5 Conclusion

This article develops a two-sector OLG model of sustainable growth by incorporating a polluting production sector and environmental preferences into the classical Diamond (1965) framework. The resulting model is analytically tractable, accommodates capital-intensity reversals, and provides a framework for studying the role of fiscal policy for the decarbonisation of an economy. Our welfare analysis highlights that the optimal extent of pollution reduction depends on the technologies, discounting, and agents' subjective valuation of the externality. The welfare-maximising allocation, including modified golden-rule steady states, can be decentralised using an emissions tax and intergenerational transfers. The emissions tax, which is equal to the sum of the discounted marginal future damages, implements the optimal pollution levels. In a modified golden-rule steady state, it accounts for the trade-off between steady-state consumption and pollution. The receipts from the emissions tax are distributed to agents to compensate them for the policy-induced reduction in factor incomes. The intergenerational transfers enable the government to choose the allocation of the compensation payments and, at the same time, implement efficient capital accumulation.

The possible coexistence of multiple modified golden-rule steady states suggests that the model may exhibit complex dynamics. An analysis of the steady states' stability properties and the qualitative dynamics arising from optimal allocations is outside the scope of this article. In view of implementing optimal allocations, the stability question should be addressed in future research. Another interesting avenue for future research would be to incorporate income and wealth inequality into the model and explore how the compensation payments can be allocated in a socially just manner. Limitations stem from the assumption that agents' consumption and environmental preferences are additive-separable, as the utility of consumption may well depend on the state of the environment. Addressing this interdependence by adopting a more general class of preferences could yield valuable insights.

A Appendix

Proof of Lemma 1. The proof is given in Ritschel and Wenzelburger (2024). □

Proof of Proposition 1. Let $(k_0, e_0) \in \mathbb{R}_{++} \times \mathbb{R}_{+}$ be arbitrary but fixed.

Step 1 (Upper bound). We show that for each feasible allocation $\big\{(k_t, e_t, y_t^b, c_t^1, c_t^2)\big\}_{t=0}^{\infty} \in \Pi(k_0, e_0)$, the social welfare $\mathcal{W}\big(\big\{(k_t, e_t, y_t^b, c_t^1, c_t^2)\big\}_{t=0}^{\infty}\big)$ is bounded from above. Observe that for each $k \geq 0$ and each $(y^b, c^1, c^2) \in Q(k)$, we have

$$A(k, y^b, c^1, c^2) \leq \tfrac{1}{1+n}\big[(1-\delta)k + f_b(k) + f_g(k)\big]. \tag{A.1}$$

It follows from (A.1) and Assumption 1 that every sequence $\{k_t\}_{t=0}^{\infty}$ starting from some $(k_0, e_0) \in \mathbb{R}_{++} \times \mathbb{R}_{+}$ that is generated recursively by the first equation in (3.3) is bounded. The sequence $\{e_t\}_{t=0}^{\infty}$ associated with $\{k_t\}_{t=0}^{\infty}$ generated by the second equation in (3.3) is also bounded. Therefore, we can conclude that the sequence $\big\{(k_t, e_t, y_t^b, c_t^1, c_t^2)\big\}_{t=0}^{\infty}$ is bounded, implying that the one-period return function is bounded from above,

$$g(k_t, e_t, y_t^b, c_t^1, c_t^2) < \infty \quad \text{for all } t \geq 0.$$

Since $\gamma \in (0,1)$, the infinite sum of the discounted welfare levels converges, so that

$$W\big(\big\{(k_t, e_t, y_t^b, c_t^1, c_t^2)\big\}_{t=0}^{\infty}\big) < \infty.$$

Hence, the social welfare is bounded from above.

Step 2 (Lower bound). The existence of a feasible allocation $\big\{(k_t, e_t, y_t^b, c_t^1, c_t^2)\big\}_{t=0}^{\infty} \in \Pi(k_0, e_0)$ for which the social welfare attains a value larger than $-\infty$ is established. For each $t \geq 0$, choose

$$y_t^b = 0 \quad \text{and} \quad c_t^1 = \tfrac{c_t^2}{1+n} = \tfrac{1}{2}\big[f_g(k_t) - (n+\delta)k_t\big].$$

For this choice, only the green sector produces so that

$$\begin{aligned} k_{t+1} &= A(k_t, 0, c_t^1, c_t^2) = \tfrac{1}{1+n}\Big[f_g(k_t) + (1-\delta)k_t - c_t^1 - \tfrac{c_t^2}{1+n}\Big] = k_t \\ e_{t+1} &= E(e_t, 0) = \tfrac{1-\zeta}{1+n} e_t \end{aligned}$$

for all times $t \geq 0$. Clearly, the social welfare $\mathcal{W}\big(\big\{(k_0, e_t, 0, c_0^1, c_0^2)\big\}_{t=0}^{\infty}\big)$ of the resulting feasible allocation $\big\{(k_0, e_t, 0, c_0^1, c_0^2)\big\}_{t=0}^{\infty} \in \Pi(k_0, e_0)$ is finite.

Step 3 (Value function). Step 1 and Step 2 combined imply that the supremum

$$\sup\Big\{\mathcal{W}\big(\{(k_t, e_t, y_t^b, c_t^1, c_t^2)\}_{t=0}^{\infty}\big) \;\Big|\; \{(k_t, e_t, y_t^b, c_t^1, c_t^2)\}_{t=0}^{\infty} \in \Pi(k_0, e_0)\Big\}$$

exists and is finite. If the supremum exists, then it is unique. Since $(k_0, e_0) \in \mathbb{R}_{++} \times \mathbb{R}_{+}$ were arbitrary, the value function $\mathcal{V}$ is well defined on $\mathbb{R}_{++} \times \mathbb{R}_{+}$. □

Proof of Lemma 2. *Step 1 (Bellman equation).* Let $(k_0, e_0) \in \mathbb{R}_{++} \times \mathbb{R}_+$ be arbitrary but fixed. For any feasible allocation $\{(k_t, e_t, y_t^b, c_t^1, c_t^2)\}_{t=0}^{\infty} \in \Pi(k_0, e_0)$ generated by the maps A and E, we have

$$
\begin{aligned}
&\sum_{t=0}^{\infty} \gamma^t g(k_t, e_t, y_t^b, c_t^1, c_t^2) && \text{(A.2)}\\
&= g(k_0, e_0, y_0^b, c_0^1, c_0^2) + \gamma \sum_{t=0}^{\infty} \gamma^t g(k_{t+1}, e_{t+1}, y_{t+1}^b, c_{t+1}^1, c_{t+1}^2)\\
&\leq g(k_0, e_0, y_0^b, c_0^1, c_0^2) + \gamma \mathcal{V}(k_1, e_1)\\
&\leq \sup\Big\{ g(k_0, e_0, y^b, c^1, c^2) + \gamma \mathcal{V}\big(A(k_0, y^b, c^1, c^2), E(e_0, y^b)\big) \;\Big|\; (y^b, c^1, c^2) \in Q(k_0) \Big\}.
\end{aligned}
$$

The claim now follows from the definition of the supremum: $\mathcal{V}(k_0, e_0)$ is by definition the smallest upper bound of (A.2) and, analogously, $\mathcal{V}(k_1, e_1)$ the smallest upper bound for all feasible allocations in $\Pi(k_1, e_1)$, where

$$
k_1 = A(k_0, y_0^b, c_0^1, c_0^2) \quad \text{and} \quad e_1 = E(e_0, y_0^b).
$$

Step 2 (Concavity and continuity of $\mathcal{V}$). We first show that any convex combination of any two paths $\{(k_t^j, e_t^j)\}_{t=0}^{\infty}$, $j = 1, 2$, associated with two feasible allocations may be generated by a feasible allocation. To this end, let $\lambda \in [0, 1]$ be arbitrary but fixed and for each $t \geq 0$, set

$$
\begin{aligned}
k_t^\lambda &= \lambda k_t^1 + (1 - \lambda) k_t^2\\
e_t^\lambda &= \lambda e_t^1 + (1 - \lambda) e_t^2.
\end{aligned}
$$

We will show that for each $t \geq 0$, there exists $q_t^\lambda = (y_t^{b\lambda}, c_t^{1\lambda}, c_t^{2\lambda}) \in Q(k_t^\lambda)$ such that

$$
\begin{aligned}
k_{t+1}^\lambda &= A(k_t^\lambda, q_t^\lambda) && \text{(A.3)}\\
e_{t+1}^\lambda &= E(e_t^\lambda, y_t^{b\lambda}) = \tfrac{1}{1+n}\big[(1 - \zeta) e_t^\lambda + \epsilon y_t^{b\lambda}\big].
\end{aligned}
$$

It follows from Lemma 1 that A is a concave function. Hence,

$$
k_{t+1}^\lambda \leq A(k_t^\lambda, \lambda q_t^1 + (1 - \lambda) q_t^2), \tag{A.4}
$$

and the convexity of the set $Q(k_t^\lambda)$ implies that $\lambda q_t^1 + (1 - \lambda) q_t^2 \in Q(k_t^\lambda)$, where $q_t^j = (y_t^{bj}, c_t^{1j}, c_t^{2j}) \in Q(k_t^j)$, $j = 1, 2$, are the policies in period t that belong to the respective feasible allocations.

Since E is linear, we have $y_t^{b\lambda} = \lambda y_t^{b1} + (1 - \lambda) y_t^{b2}$. Given $y_t^{b\lambda}$, it follows from (A.4) that there exists a consumption bundle $(c_t^{1\lambda}, c_t^{2\lambda})$ such that $q_t^\lambda = (y_t^{b\lambda}, c_t^{1\lambda}, c_t^{2\lambda}) \in Q(k_t^\lambda)$ and

(A.3) holds. As a consequence, $\{(k_t^\lambda, e_t^\lambda, q_t^\lambda)\}_{t=0}^\infty \in \Pi(k_0^\lambda, e_0^\lambda)$. Since

$$c_t^{1\lambda} + \tfrac{c_t^{2\lambda}}{1+n} > \lambda\left(c_t^{11} + \tfrac{c_t^{21}}{1+n}\right) + (1-\lambda)\left(c_t^{12} + \tfrac{c_t^{22}}{1+n}\right),$$

the consumption bundle $(c_t^{1\lambda}, c_t^{2\lambda})$ may be chosen such that

$$g(k_t^\lambda, e_t^\lambda, q_t^\lambda) > \lambda g(k_t^1, e_t^1, q_t^1) + (1-\lambda) g(k_t^2, e_t^2, q_t^2). \tag{A.5}$$

The concavity of the value function $\mathcal{V}$ is now established as follows. By definition of $\mathcal{V}$, for any $\varepsilon > 0$, there exist a feasible allocation $\big\{(k_t^j, e_t^j, q_t^j)\big\}_{t=0}^\infty \in \Pi(k_0^j, e_0^j)$, $j = 1, 2$, such that

$$\sum_{t=0}^\infty \gamma^t g(k_t^j, e_t^j, q_t^j) > \mathcal{V}(k_0^j, e_0^j) - \varepsilon. \tag{A.6}$$

Since $\big\{(k_t^\lambda, e_t^\lambda, q_t^\lambda)\big\}_{t=0}^\infty \in \Pi(k_0^\lambda, e_0^\lambda)$, it follows from (A.5) that

$$\mathcal{V}(k_0^\lambda, e_0^\lambda) \geq \sum_{t=0}^\infty \gamma^t g(k_t^\lambda, e_t^\lambda, q_t^\lambda) \geq \lambda \sum_{t=0}^\infty \gamma^t g(k_t^1, e_t^1, q_t^1) + (1-\lambda) \sum_{t=0}^\infty \gamma^t g(k_t^2, e_t^2, q_t^2).$$

Taking suprema, (A.6) implies

$$\mathcal{V}(k_0^\lambda, e_0^\lambda) \geq \lambda \mathcal{V}(k_0^1, e_0^1) + (1-\lambda)\mathcal{V}(k_0^2, e_0^2) - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this shows that $\mathcal{V}$ is concave. Concave functions defined on open subsets of $\mathbb{R}^2$ are continuous, e.g., see Rockafellar (1970).

Step 3 (Differentiability of $\mathcal{V}$).

Let $(k_0, e_0) \in \mathbb{R}_{++} \times \mathbb{R}_+$ be arbitrary but fixed. We will show in the proof of Theorem 1 that there exists an optimal policy $(y_0^b, c_0^1, c_0^2) \in Q(k_0)$ such that

$$\mathcal{V}(k_0, e_0) = g(k_0, e_0, y_0^b, c_0^1, c_0^2) + \gamma \mathcal{V}\big(A(k_0, y_0^b, c_0^1, c_0^2), E(e_0, y_0^b)\big). \tag{A.7}$$

Given $e_1 = E(e_0, y_0^b)$, the function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$, defined by

$$\Phi(e) := \tfrac{1+n}{\epsilon} e_1 - \tfrac{1-\zeta}{\epsilon} e,$$

satisfies

$$E(e, \Phi(e)) = e_1 = E(e_0, y_0^b) \quad \text{for all } \; e \in \mathbb{R}_+. \tag{A.8}$$

Choose an open neighbourhood $\mathscr{U}(k_0, e_0)$ of (k_0, e_0) and set

$$c_1(k, e) = f(k, \Phi(e)) - f(k_0, y_0^b) + c_0^1 \quad \text{and} \quad c_2(k, e) \equiv c_0^2,$$

so that

$$A(k, \Phi(e), c_1(k, e), c_0^2) = A(k_0, y_0^b, c_0^1, c_0^2) = k_1$$

for all $(k,e) \in \mathscr{U}(k_0,e_0)$. Observe that

$$y_0^b = \Phi(e_0), \quad c_0^1 = c_1(k_0,e_0), \quad \text{and} \quad c_0^2 = c_2(k_0,e_0). \tag{A.9}$$

From (A.7), we can infer that for each $(k,e) \in \mathscr{U}(k_0,e_0)$,

$$\mathcal{V}(k,e) \geq g(k,e,\Phi(e),c_1(k,e),c_0^2) + \gamma\mathcal{V}(k_1,e_1)$$

and thus

$$\mathcal{V}(k,e) - \mathcal{V}(k_0,e_0) \geq g(k,e,\Phi(e),c_1(k,e),c_0^2) - g(k_0,e_0,y_0^b,c_0^1,c_0^2).$$

This shows that the concave function $\mathcal{V}$ is bounded from below on $\mathscr{U}(k_0,e_0)$ by a differentiable function. As a consequence, $\mathcal{V}$ is differentiable at (k_0,e_0). Since (k_0,e_0) was arbitrary, it follows that $\mathcal{V}$ is differentiable on $\mathbb{R}_{++} \times \mathbb{R}_+$. □

Proof of Theorem 1. *Step 1 (Existence of policy functions).* Without loss of generality, we may consider period $t = 0$. Let $(k_0,e_0) \in \mathbb{R}_{++} \times \mathbb{R}_+$ be arbitrary but fixed and define the set

$$\mathscr{Q}(k_0,e_0) := \Big\{(y^b,c^1,c^2,k,e) \in \mathbb{R}_+^5 \;\Big|\; (y^b,c^1,c^2) \in Q(k_0),\ k = A(k_0,y^b,c^1,c^2) \text{ and } e = E(e_0,y^b)\Big\}. \tag{A.10}$$

Since each set $Q(k_0)$ is convex, each $A(k_0,\cdot)$ is concave, and each $E(e_0,\cdot)$ is linear, each set $\mathscr{Q}(k_0,e_0)$ is concave. Applying the Bellman equation twice, Lemma 2 implies

$$\mathcal{V}(k_0,e_0) = \sup\Big\{g(k_0,e_0,y_0^b,c_0^1,c_0^2) + \gamma g(k_1,e_1,y_1^b,c_1^1,c_1^2) + \gamma^2\mathcal{V}(k_2,e_2) \;\Big|\; (y_0^b,c_0^1,c_0^2,k_1,e_1) \in \mathscr{Q}(k_0,e_0) \text{ and } (y_1^b,c_1^1,c_1^2,k_2,e_2) \in \mathscr{Q}(k_1,e_1)\Big\}. \tag{A.11}$$

We show that there exists $\delta_0 > 0$ such that the supremum in (A.11) is not attained in any

$$(y_0^b,c_0^1,c_0^2,k_1,e_1) \in \mathscr{Q}(k_0,e_0) \quad \text{with either } c_0^1 = 0 \text{ or } c_0^2 = 0 \text{ or } k_1 < \delta_0.$$

Set

$$k_2 = \tfrac{1}{1+n}\Big[f(k_1,y_1^b) - \big(c_1^1 + \tfrac{c_1^2}{1+n}\big)\Big] \quad \text{and} \quad \hat{k}_2 = \tfrac{1}{1+n}\Big[f(\hat{k}_1,\hat{y}_1^b) - \big(\hat{c}_1^1 + \tfrac{\hat{c}_1^2}{1+n}\big)\Big]. \tag{A.12}$$

It follows from (A.12) that $k_2 = \hat{k}_2$ if and only if

$$f(k_1,y_1^b) - f(\hat{k}_1,\hat{y}_1^b) = \big(c_1^1 + \tfrac{c_1^2}{1+n}\big) - \big(\hat{c}_1^1 + \tfrac{\hat{c}_1^2}{1+n}\big). \tag{A.13}$$

Consider $(y_0^b,0,c_0^2,k_1,e_1) \in \mathscr{Q}(k_0,e_0)$. Choose $\varepsilon_0 > 0$ and $(\hat{y}_0^b,\hat{c}_0^1,\hat{c}_0^2) = (y_0^b,\varepsilon_0(1+n),c_0^2) \in$

$Q(k_0)$ so that

$$\hat{k}_1 = A(k_0, \hat{y}_0^b, \hat{c}_0^1, \hat{c}_0^2) = k_1 - \varepsilon_0 \quad \text{and} \quad \hat{e}_1 = e_1.$$

Next, choose $(\hat{y}_1^b, \hat{c}_1^1, \hat{c}_1^2) \in Q(\hat{k}_1)$ with $\hat{c}_1^1 < c_1^1$, $\hat{c}_1^2 = c_1^2$ such that (A.13) holds. Since $\hat{y}_1^b \leq y_1^b$, it follows that $k_2 = \hat{k}_2$ and $\hat{e}_2 = E(\hat{e}_1, \hat{y}_1^b) \leq E(e_1, y_1^b) = e_2$ and thus $\mathcal{V}(\hat{k}_2, \hat{e}_2) \geq \mathcal{V}(k_2, e_2)$. Hence, if $\varepsilon_0 > 0$ is sufficiently small, the Inada condition on u implies that $(y_0^b, \varepsilon_0(1+n), c_0^2, \hat{k}_1, \hat{e}_1) \in \mathscr{Q}(k_0, e_0)$ attains higher level of utility than $(y_0^b, 0, c_0^2, k_1, e_1)$. Hence, the supremum cannot be attained with $c_0^1 = 0$. An analogous argument applies for the case with $c_0^2 = 0$.

Consider now $(y_0^b, c_0^1, c_0^2, k_1, e_1) \in \mathscr{Q}(k_0, e_0)$ with $0 < k_1 < \delta_0$. Choose $\varepsilon_0, \epsilon_1 > 0$ and

$$(\hat{y}_0^b, \hat{c}_0^1, \hat{c}_0^2) = (y_0^b, c_0^1 - \varepsilon_0, c_0^2) \in Q(k_0) \quad \text{and} \quad (\hat{y}_1^b, \hat{c}_1^1, \hat{c}_1^2) = (y_1^b, c_1^1 + \varepsilon_1, c_1^2) \in Q(\hat{k}_1)$$

so that $\hat{k}_1 > k_1$ and $\hat{k}_2 = k_2$ holds via (A.13). Since $\hat{y}_j^b = y_j^b$, $j = 0, 1$, it follows that $\hat{e}_1 = e_1$ and $\hat{e}_2 = e_2$. Since $f(0,0) = 0$, it follows that c_1^1 becomes arbitrarily small if k_1 is arbitrarily small. Hence, if $\delta_0 > 0$ is sufficiently small, the Inada condition on u implies that $(y_0^b, \hat{c}_0^1, c_0^2, \hat{k}_1, \hat{e}_1) \in \mathscr{Q}(k_0, e_0)$ attains higher level of utility than $(y_0^b, c_0^1, c_0^2, k_1, e_1)$.

Since the objective function $g + \gamma\mathcal{V}$ in the Bellman equation is strictly concave and continuous on any compact subset of $\mathscr{Q}(k_0, e_0)$, its maximum is uniquely determined and attained in the interior of $\mathscr{Q}(k_0, e_0)$. For any given $(k_0, e_0) \in \mathbb{R}_{++} \times \mathbb{R}_+$, the Bellman equation thus reads

$$\mathcal{V}(k_0, e_0) = \max\Big\{ g(k_0, e_0, y_0^b, c_0^1, c_0^2) + \gamma\mathcal{V}(k_1, e_1) \;\Big|\; (y_0^b, c_0^1, c_0^2, k_1, e_1) \in \mathscr{Q}(k_0, e_0) \Big\}. \quad \text{(A.14)}$$

The maximiser $(y_0^{b*}, c_0^{1*}, c_0^{2*})$ is uniquely determined and it depends only on (k_0, e_0). Therefore, there exist optimal policy functions $y_*^b : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}_+$ and $c_*^1, c_*^2 : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}_{++}$ with $y_0^{b*} = y_*^b(k_0, e_0) \in [0, f_b(k_0)]$, $c_0^{1*} = c_*^1(k_0, e_0) > 0$, and $c_0^{2*} = c_*^2(k_0, e_0) > 0$.

Step 2 (Existence and uniqueness of the optimal allocation). Since the policy functions y_*^b, c_*^1, c_*^2 are well defined on $\mathbb{R}_{++} \times \mathbb{R}_+$, it follows that for any given initial condition $(k_0, e_0) \in \mathbb{R}_{++} \times \mathbb{R}_+$, there exists a uniquely determined optimal allocation

$$\big\{(k_t^*, e_t^*, y_t^{b*}, c_t^{1*}, c_t^{2*})\big\}_{t=0}^{\infty} \in \Pi(k_0, e_0),$$

which is recursively defined by

$$\begin{aligned} k_{t+1}^* &= A\big(k_t^*, y_*^b(k_t^*, e_t^*), c_*^1(k_t^*, e_t^*), c_*^2(k_t^*, e_t^*)\big) \\ e_{t+1}^* &= E\big(e_t^*, y_*^b(k_t^*, e_t^*)\big) \end{aligned}$$

with $(k_0^*, e_0^*) = (k_0, e_0)$.

Step 3 (First-order conditions). For better readability, we omit the $*$ superscript indicating the optimal allocation. Setting

$$V(k_0, e_0, y^b, c^1, c^2) = g(k_0, e_0, y^b, c^1, c^2) + \mathcal{V}\big(A(k_0, y^b, c^1, c^2), E(e_0, y^b)\big),$$

the Lagrangian for Problem (A.14) is

$$\mathcal{L}(y^b, c^1, c^2, \lambda_0^0, \lambda_0^1, \lambda_0^2) = V(k_0, e_0, y^b, c^1, c^2) + \lambda_0^0\big[f(k_0, y^b) - c^1 - \tfrac{c^2}{1+n}\big] + \lambda_0^1 y^b + \lambda_0^2\big[f_b(k_0) - y^b\big].$$

By Lemma 2, the value function $\mathcal{V}$ is differentiable. Setting

$$k_1 = A(k_0, y_0^b, c_0^1, c_0^2) \quad \text{and} \quad e_1 = E(e_0, y_0^b) = \tfrac{z_0}{1+n},$$

then differentiation of $\mathcal{L}$ w.r.t. (y^b, c^1, c^2) shows that the first-order conditions for the optimal policy $(y_0^b, c_0^1, c_0^2) \in Q(k_0)$ are

$$-(1 + \tfrac{\beta}{\gamma})\epsilon\mu'(z_0) + \tfrac{\gamma}{1+n}\frac{\partial \mathcal{V}}{\partial k}(k_1, e_1)\frac{\partial f}{\partial y^b}(k_0, y_0^b) + \tfrac{\gamma}{1+n}\frac{\partial \mathcal{V}}{\partial e}(k_1, e_1)\epsilon + \lambda_0^1 - \lambda_0^2 \stackrel{!}{=} 0 \tag{A.15}$$

$$u'(c_0^1) - \tfrac{\gamma}{1+n}\frac{\partial \mathcal{V}}{\partial k}(k_1, e_1) \stackrel{!}{=} \lambda_0^0 \tag{A.16}$$

$$\tfrac{\beta}{\gamma}u'(c_0^2) - \tfrac{\gamma}{(1+n)^2}\frac{\partial \mathcal{V}}{\partial k}(k_1, e_1) \stackrel{!}{=} \tfrac{\lambda_0^0}{1+n}. \tag{A.17}$$

The complementary slackness conditions for Problem (A.14) are

$$\lambda_0^0\left[f(k_0, y_0^b) - c_0^1 - \tfrac{c_0^2}{1+n}\right] \stackrel{!}{=} 0 \tag{A.18}$$

$$\lambda_0^1 y_0^b \stackrel{!}{=} 0 \tag{A.19}$$

$$\lambda_0^2\big[f_b(k_0) - y_0^b\big] \stackrel{!}{=} 0 \tag{A.20}$$

and the non-negativity conditions are $\lambda_0^0, \lambda_0^1, \lambda_0^2 \geq 0$. As we have shown in Step 1, the optimal policy (y_0^b, c_0^1, c_0^2) satisfies $k_1 = A(k_0, y_0^b, c_0^1, c_0^2) > 0$. Consequently, the complementary slackness condition (A.18) holds with $\lambda_0^0 = 0$. Conditions (A.16) and (A.17), therefore, become

$$\frac{u'(c_0^1)}{\beta u'(c_0^2)} \stackrel{!}{=} \tfrac{1+n}{\gamma} \tag{A.21}$$

$$u'(c_0^1) \stackrel{!}{=} \tfrac{\gamma}{1+n}\frac{\partial \mathcal{V}}{\partial k}(k_1, e_1). \tag{A.22}$$

Equation (A.21) is the first-order condition (3.7) given in the theorem. By the Envelope theorem (e.g., see Mas-Colell, Whinston, & Green, 1995, p. 965), we have

$$\frac{\partial V}{\partial k}(k_0, e_0, y_0^b, c_0^1, c_0^2) = \tfrac{\gamma}{1+n}\frac{\partial \mathcal{V}}{\partial k}(k_1, e_1)\frac{\partial f}{\partial k}(k_0, y_0^b) + \lambda_0^2 f_b'(k_0). \tag{A.23}$$

Moreover, the Bellman equation implies that the derivatives of the objective function V

and the value function $\mathcal{V}$ coincide, so that

$$\frac{\partial \mathcal{V}}{\partial k}(k_0, e_0) = \frac{\partial V}{\partial k}(k_0, e_0, y_0^b, c_0^1, c_0^2). \tag{A.24}$$

It follows from (A.22) – (A.24) that

$$\frac{\partial \mathcal{V}}{\partial k}(k_0, e_0) = u'(c_0^1)\Big[\frac{\partial f}{\partial k}(k_0, y_0^b) + \tfrac{\lambda_0^2}{u'(c_0^1)} f_b'(k_0)\Big]. \tag{A.25}$$

Inserting (A.25) and (A.21), Condition (A.22) then takes the form

$$\frac{u'(c_0^1)}{\beta u'(c_1^2)} \stackrel{!}{=} \frac{\partial f}{\partial k}(k_1, y_1^b) + \tfrac{\lambda_1^2}{u'(c_1^1)} f_b'(k_1). \tag{A.26}$$

Re-normalising the Lagrange multiplier, then (A.26) is the first-order condition (3.8) stated in the theorem.

By the same vein, the Envelope theorem and the Bellman equation imply

$$\begin{aligned}\frac{\partial \mathcal{V}}{\partial e}(k_0, e_0) &= \frac{\partial g}{\partial e}(k_0, e_0, y_0^b, c_0^1, c_0^2) + \tfrac{\gamma(1-\zeta)}{1+n}\frac{\partial \mathcal{V}}{\partial e}(k_1, e_1)\\ &= -\big(1 + \tfrac{\beta}{\gamma}\big)\mu'(z_0)(1-\zeta) + \tfrac{\gamma(1-\zeta)}{1+n}\frac{\partial \mathcal{V}}{\partial e}(k_1, e_1).\end{aligned} \tag{A.27}$$

It follows from (A.15), (A.22), and (A.27) that

$$\frac{\partial \mathcal{V}}{\partial e}(k_0, e_0) = -u'(c_0^1)\tfrac{(1-\zeta)}{\epsilon}\Big[\frac{\partial f}{\partial y^b}(k_0, y_0^b) + \tfrac{\lambda_0^1 - \lambda_0^2}{u'(c_0^1)}\Big]. \tag{A.28}$$

Inserting (A.25) and (A.28) into (A.15) then yields

$$\tfrac{\gamma(1-\zeta)}{1+n}\frac{u'(c_1^1)}{u'(c_0^1)}\left[\frac{\partial f}{\partial y^b}(k_1, y_1^b) + \tfrac{\lambda_1^1 - \lambda_1^2}{u'(c_1^1)}\right] + \epsilon(1 + \tfrac{\beta}{\gamma})\frac{\mu'(z_0)}{u'(c_0^1)} \stackrel{!}{=} \frac{\partial f}{\partial y^b}(k_0, y_0^b) + \tfrac{\lambda_0^1 - \lambda_0^2}{u'(c_0^1)}. \tag{A.29}$$

Re-normalising the shadow prices and recursively applying (A.29) then establishes the first-order condition (3.9) given in the theorem. □

Proof of Corollary 1. Let $(k_t, e_t) \in \mathbb{R}_{++} \times \mathbb{R}_+$ be arbitrary but fixed.

Part (i). Setting $\zeta = 1$ and

$$\Psi(k_t, e_t, y_t^b) \coloneqq \epsilon\big(1 + \tfrac{\beta}{\gamma}\big)\frac{\mu'(\epsilon y_t^b)}{u'(c_*^1(k_t, e_t))},$$

the first-order condition (3.9) takes the form

$$\Psi(k_t, e_t, y_t^b) \stackrel{!}{=} 1 - \varrho(\Omega(k_t, y_t^{b*})) + \lambda_t^1 - \lambda_t^2. \tag{A.30}$$

There are three possible cases.

Case 1. $\lambda_t^1 > 0$ and $\lambda_t^2 = 0$. Then $y_t^{b*} = 0$ due to (3.10). Since $\Omega(k_t, 0) = \Omega_g(k_t)$, it follows from (A.30) that

$$\lambda_t^1 \geq 0 \iff \varrho(\Omega_g(k_t)) \geq 1 - \Psi(k_t, e_t, 0).$$

Case 2. $\lambda_t^1 = 0$ and $\lambda_t^2 > 0$. Then $y_t^{b*} = f_b(k_t)$ due to (3.10). Since $\Omega(k_t, f_b(k_t)) = \Omega_b(k_t)$, it follows from (A.30) that

$$\lambda_t^2 \geq 0 \iff \varrho(\Omega_b(k_t)) \leq 1 - \Psi(k_t, e_t, f_b(k_t)).$$

Case 3. $\lambda_t^1 = \lambda_t^2 = 0$. In this case, (A.30) implies that $y_t^{b*} \in [0, f_b(k_t)]$ is determined by

$$\Psi(k_t, e_t, y_t^{b*}) \stackrel{!}{=} 1 - \varrho(\Omega(k_t, y_t^{b*})).$$

Part (ii). If $\mu' \equiv 0$, then $\Psi \equiv 0$ and Cases 1 – 3 above imply that

$$y_t^{b*} = \begin{cases} 0 & \text{if } \varrho(\Omega_g(k_t)) \geq 1 \\ f_b(k_t) & \text{if } \varrho(\Omega_b(k_t)) \leq 1 \\ \text{solves } \varrho(\Omega(k_t, y_t^{b*})) = 1 & \text{otherwise} \end{cases}. \tag{A.31}$$

It remains to show that (A.31) solves the maximisation problem

$$\max_{0 \leq y^b \leq f_b(k_t)} f(k_t, y^b). \tag{A.32}$$

The Lagrangian for (A.32) is

$$\mathcal{L}(y^b, \xi_t^1, \xi_t^2) = f(k_t, y^b) + \xi_t^1 y^b + \xi_t^2 \left[f_b(k_t) - y^b\right].$$

A solution y_t^{b*} must satisfy the first-order condition

$$1 - \varrho(\Omega(k_t, y_t^{b*})) \stackrel{!}{=} \xi_t^2 - \xi_t^1 \tag{A.33}$$

and the complementary slackness conditions

$$\xi_t^1 y_t^{b*} \stackrel{!}{=} 0 \quad \text{and} \quad \xi_t^2 \left[f_b(k_t) - y_t^{b*}\right] \stackrel{!}{=} 0, \quad \xi_t^1, \xi_t^2 \geq 0.$$

These conditions are sufficient for a maximum since, by Lemma 1, the objective function f is concave. Since the first-order condition (A.33) coincides with the first-order condition (A.30) if $\psi \equiv 0$, it follows that the optimal policy y_t^{b*} is a maximiser of (A.32) if $\mu' \equiv 0$. □

Proof of Theorem 2. Using the function ψ defined in (3.17), it follows from Theorem

1 (ii) that the pair $(\bar{k}_\gamma, \bar{y}^b_\gamma)$ must satisfy the first-order conditions

$$\frac{\partial f}{\partial k}(\bar{k}, \bar{y}^b) \stackrel{!}{=} \tfrac{1+n}{\gamma} - \bar{\lambda}^2 f_b'(\bar{k}) \tag{A.34}$$
$$\frac{\partial f}{\partial y^b}(\bar{k}, \bar{y}^b) \stackrel{!}{=} \psi(\bar{k}, \bar{y}^b) - \bar{\lambda}^1 + \bar{\lambda}^2 \tag{A.35}$$

together with the complementary slackness conditions

$$\bar{\lambda}^1\, \bar{y}^b \stackrel{!}{=} 0 \quad \text{and} \quad \bar{\lambda}^2 \big[f_b(\bar{k}) - \bar{y}^b\big] \stackrel{!}{=} 0 \tag{A.36}$$

and the non-negativity conditions $\bar{\lambda}^1, \bar{\lambda}^2 \geq 0$. There are three possible cases.

Case 1. Let $\bar{\lambda}^1 > 0$ and $\bar{\lambda}^2 = 0$. Then $\bar{y}^b_\gamma = 0$ due to (A.36). Since $\Omega(\bar{k}, 0) = \Omega_g(\bar{k})$, Lemma 1 implies that (A.34) takes the form

$$f_g'(\bar{k}) \stackrel{!}{=} \tfrac{1+n}{\gamma} - 1 + \delta. \tag{A.37}$$

It follows from Assumption 1 that (A.37) admits a unique solution, which is the capital-labour ratio $\bar{k}^g$. Setting $\bar{\omega}^g = \Omega_g(\bar{k}^g)$, then (A.35) implies that $\bar{\lambda}^1 \geq 0$ if and only if

$$\varrho(\bar{\omega}^g) \geq 1 - \psi(\bar{k}^g, 0).$$

Therefore, $(\bar{k}_\gamma, \bar{y}^b_\gamma) = (\bar{k}^g, 0)$ is a steady state if $\varrho(\bar{\omega}^g) \geq 1 - \psi(\bar{k}^g, 0)$.

Case 2. Let $\bar{\lambda}^1 = \bar{\lambda}^2 = 0$. Using Lemma 1, Condition (A.35) takes the form

$$\varrho(\Omega(\bar{k}, \bar{y}^b)) \stackrel{!}{=} 1 - \psi(\bar{k}, \bar{y}^b) \tag{A.38}$$

and (A.34) the form

$$f_g'\big(\kappa_g(\Omega(\bar{k}, \bar{y}^b))\big) \stackrel{!}{=} \tfrac{1+n}{\gamma} - 1 + \delta. \tag{A.39}$$

Condition (A.39) implies that

$$\Omega(\bar{k}, \bar{y}^b) \stackrel{!}{=} \bar{\omega}^g \quad \text{and} \quad \bar{y}^b \stackrel{!}{=} \mathsf{y}_b(\bar{k}, \bar{\omega}^g). \tag{A.40}$$

Inserting (A.40) into (A.38), it follows that the capital-labour ratio is determined by

$$\varrho(\bar{\omega}^g) \stackrel{!}{=} 1 - \psi(\bar{k}, \mathsf{y}_b(\bar{k}, \bar{\omega}^g)). \tag{A.41}$$

By the intermediate-value theorem, a solution

$$\bar{k}_\gamma \in \big(\min\{\kappa_g(\bar{\omega}^g), \kappa_b(\bar{\omega}^g)\}, \max\{\kappa_g(\bar{\omega}^g), \kappa_b(\bar{\omega}^g)\}\big)$$

to (A.41) exists if either (3.18) or (3.19) holds. Given $\bar{k}_\gamma$, (A.40) implies that the brown production plan is $\bar{y}^b_\gamma = \mathsf{y}_b(\bar{k}_\gamma, \bar{\omega}^g) \in \big(0, f_b(\bar{k}_\gamma)\big)$.

Case 3. Let $\bar{\lambda}^1 = 0$ and $\bar{\lambda}^2 > 0$. Then $\bar{y}^b_\gamma = f_b(\bar{k}_\gamma)$ due to (A.36). Since $\Omega(\bar{k}, f_b(\bar{k})) =$

$\Omega_b(\bar{k})$, Lemma 1 implies that Condition (A.34) takes the form

$$\bar{\lambda}^2 \stackrel{!}{=} \frac{\frac{1+n}{\gamma} - 1 + \delta - f_g'\big(\kappa_g(\Omega_b(\bar{k}))\big)}{f_b'(\bar{k})} \tag{A.42}$$

and (A.35) the form

$$\bar{\lambda}^2 \stackrel{!}{=} 1 - \psi(\bar{k}, f_b(\bar{k})) - \varrho\big(\Omega_b(\bar{k})\big). \tag{A.43}$$

Inserting (A.42) into (A.43), it follows that the capital-labour ratio is determined by

$$f_b'(\bar{k})\big[1 - \psi(\bar{k}, f_b(\bar{k}))\big] \stackrel{!}{=} \tfrac{1+n}{\gamma} - 1 + \delta. \tag{A.44}$$

Moreover, it can be read off (A.43) that

$$\bar{\lambda}^2 \geq 0 \iff \Omega_b(\bar{k}) \geq \bar{\omega}^g \iff \bar{k} \geq \kappa_b(\bar{\omega}^g). \tag{A.45}$$

The existence of a solution to (A.44) is established using the intermediate-value theorem. Setting $\bar{k}^b = f_b'^{-1}(\frac{1+n}{\gamma} - 1 + \delta)$, we have

$$f_b'(\bar{k}^b)\big[1 - \psi(\bar{k}^b, f_b(\bar{k}^b))\big] < \tfrac{1+n}{\gamma} - 1 + \delta.$$

Since $f_j'' < 0$ and $\kappa_j' > 0$, $j = g, b$,

$$\varrho(\bar{\omega}^g) \;=\; \frac{\frac{1+n}{\gamma} - 1 + \delta}{f_b'(\kappa_b(\bar{\omega}^g))} = \frac{f_b'\big(\kappa_b(\Omega_b(\bar{k}^b))\big)}{f_b'\big(\kappa_b(\Omega_g(\bar{k}^g))\big)} < 1 \iff \Omega_g(\bar{k}^g) < \Omega_b(\bar{k}^b) \iff \kappa_b(\bar{\omega}^g) < \bar{k}^b.$$

On the other hand,

$$f_b'(\kappa_b(\bar{\omega}^g))\big[1 - \psi\big(\kappa_b(\bar{\omega}^g), f_b(\kappa_b(\bar{\omega}^g))\big)\big] \geq \tfrac{1+n}{\gamma} - 1 + \delta$$

if and only if

$$1 - \psi\big(\kappa_b(\bar{\omega}^g), f_b(\kappa_b(\bar{\omega}^g))\big) \geq \varrho(\bar{\omega}^g). \tag{A.46}$$

Therefore, if (A.46) holds, then there exists a solution $\bar{k}_\gamma \in [\kappa_b(\bar{\omega}^g), \bar{k}^b)$ to (A.44) such that $\bar{k}_\gamma$ together with $y_\gamma^b = f_b(\bar{k}_\gamma)$ is a steady state. □

Proof of Corollary 2. The first claim is included by setting $\psi \equiv 0$ in the proof of Theorem 2. The second claim obtains as follows. The Lagrangian is

$$\mathcal{L}(\bar{k}, \bar{y}^b, \lambda_1, \lambda_2) = \phi(\bar{k}, \bar{y}^b) + \lambda_1 \bar{y}^b + \lambda_2\big[f_b(\bar{k}) - \bar{y}^b\big].$$

The first-order conditions are

$$\lambda_2 f_b'(\bar{k}) = n + \delta - f_g'\big(\kappa_g(\Omega(\bar{k}, \bar{y}^b))\big) \tag{A.47}$$

$$1 = \varrho(\Omega(\bar{k}, \bar{y}^b)) + \lambda_2 - \lambda_1 \tag{A.48}$$

and the complementary slackness conditions are

$$\lambda_1 \bar{y}^b = 0 \quad \text{and} \quad \lambda_2 \big[f_b(\bar{k}) - \bar{y}^b \big] = 0, \quad \lambda_1, \lambda_2 \geq 0 \tag{A.49}$$

There are three cases.

Case 1. $\lambda_1 > 0$ and $\lambda_2 = 0$. Then (A.49) implies $\bar{y}^b = 0$ and, since $\Omega(\bar{k}, 0) = \Omega_g(\bar{k})$, (A.47) takes the form

$$f_g'(\bar{k}) = n + \delta. \tag{A.50}$$

Assumption 1 implies that (A.50) admits a unique solution $0 < \bar{k}^g < \infty$. The first-order condition (A.48) takes the form

$$\lambda_1 = \varrho(\bar{\omega}^g) - 1,$$

so that $\lambda_1 \geq 0$ if and only if $\varrho(\bar{\omega}^g) \geq 1$. Moreover, $\phi(\bar{k}^g, 0) = w_g(\bar{k}^g)$.

Case 2. $\lambda_1 = 0$ and $\lambda_2 > 0$. Then (A.49) implies $\bar{y}^b = f_b(\bar{k})$. Since $\Omega(\bar{k}, f_b(\bar{k})) = \Omega_b(\bar{k})$, (A.47) becomes

$$\lambda_2 \, f_b'\big(\kappa_b(\Omega_b(\bar{k}))\big) = n + \delta - f_g'\big(\kappa_g(\Omega_b(\bar{k}))\big) \tag{A.51}$$

and (A.48) becomes

$$\lambda_2 = 1 - \varrho(\Omega_b(\bar{k})). \tag{A.52}$$

Inserting (A.52) into (A.51), it follows that

$$f_b'(\bar{k}) = n + \delta. \tag{A.53}$$

Assumption 1 implies that (A.53) admits a unique solution $0 < \bar{k}^b < \infty$. As shown above,

$$\lambda_2 \geq 0 \iff \varrho(\Omega_b(\bar{k}^b)) \leq 1 \iff \varrho(\bar{\omega}^g) \leq 1.$$

Moreover, $\phi(\bar{k}^b, f_b(\bar{k}^b)) = w_b(\bar{k}^b)$.

Case 3. $\lambda_1 = \lambda_2 = 0$. Then (A.47) and (A.48) take the form

$$f_g'\big(\kappa_g(\Omega(\bar{k}, \bar{y}^b))\big) = n + \delta \tag{A.54}$$

$$\varrho(\Omega(\bar{k}, \bar{y}^b)) = 1. \tag{A.55}$$

Condition (A.54) implies $\Omega(\bar{k}, \bar{y}^b) = \bar{\omega}^g$, so that (A.55) implies $\varrho(\bar{\omega}^g) = 1$. In this case, $w_g(\bar{k}^g) = w_b(\bar{k}_b)$ so that ϕ attains the same maximum in either boundary allocation $(\bar{k}^g, 0)$ and $(\bar{k}^b, f_b(\bar{k}^b))$. □

Proof of Proposition 2. Let $(k_t, e_t) \in \mathbb{R}_{++} \times \mathbb{R}_+$ and $\eta_t = \eta(k_t, e_t) \in \mathbb{R}$ be arbitrary but fixed. Since capital and labour are perfectly mobile and paid their marginal products, an interior temporary equilibrium in which both sectors are producing obtains for a wage-rental ratio $\omega_t \in \big[\Omega_{\min}(k_t), \Omega_{\max}(k_t)\big]$ such that

$$\varrho(\omega_t) \overset{!}{=} 1 - \epsilon\eta_t. \tag{A.56}$$

The concavity of the production-possibility frontier implies that $\varrho(\Omega_g(k_t)) \leq \varrho(\Omega_b(k_t))$, so that a solution $\omega_t \in \big[\Omega_{\min}(k_t), \Omega_{\max}(k_t)\big]$ to (A.56) exists if and only if $\varrho(\Omega_g(k_t)) \leq 1-\epsilon\eta_t \leq \varrho(\Omega_b(k_t))$. Since ϱ is either monotonically increasing or decreasing on the interval in question, the equilibrium wage-rental is uniquely determined by $\omega_t = \varrho^{-1}(1-\epsilon\eta_t)$.

A solution to (A.56) does not exist if either $1-\epsilon\eta_t < \varrho(\Omega_g(k_t))$ or $\varrho(\Omega_b(k_t)) < 1-\epsilon\eta_t$. In either case, a boundary temporary equilibrium is obtained. In the first case, we have

$$f_g'(\kappa_g(\omega)) > [1-\epsilon\eta_t] f_b'(\kappa_b(\omega)) \quad \text{and} \quad w_g(\kappa_g(\omega)) > [1-\epsilon\eta_t] w_b(\kappa_b(\omega))$$

for all $\omega \in \big[\Omega_{\min}(k_t), \Omega_{\max}(k_t)\big]$, so that the green sector receives the entire capital and labour. Hence, the equilibrium wage-rental ratio is $\omega_t = \Omega_g(k_t)$. In the second case,

$$f_g'(\kappa_g(\omega)) < [1-\epsilon\eta_t] f_b'(\kappa_b(\omega)) \quad \text{and} \quad w_g(\kappa_g(\omega)) < [1-\epsilon\eta_t] w_b(\kappa_b(\omega))$$

for all $\omega \in \big[\Omega_{\min}(k_t), \Omega_{\max}(k_t)\big]$, so that the brown sector receives the entire capital and labour. Hence, the equilibrium wage-rental ratio is $\omega_t = \Omega_b(k_t)$.

Hence, for any given $(k_t, e_t) \in \mathbb{R}_{++} \times \mathbb{R}_+$ and $\eta_t = \eta(k_t, e_t) \in \mathbb{R}$, the equilibrium wage-rental ratio is uniquely determined by $\omega_t = \Omega_{\text{eq}}(k_t, \eta_t)$, with the function Ω_{eq} as defined in the proposition. □

Proof of Lemma 3. Setting $e_{t+1} = E_{\text{eq}}(k_t, e_t)$, Condition (4.9) is equivalent to

$$\tau(k_t^e, e_{t+1}) \geq -\frac{\frac{1}{1+n} R(k_t^e, e_{t+1}) w^d(k_t, e_t) + \eta(k_t^e, e_{t+1}) \epsilon\, \mathsf{y}_{\text{eq}}^b(k_t^e, e_{t+1})}{w(k_t^e, e_{t+1})}. \tag{A.57}$$

Since the tax-policy rules η, τ define a feasible fiscal policy in the sense of Definition 2, they satisfy

$$\tau(k_t^e, e_{t+1}) \geq -\frac{R(k_t^e, e_{t+1}) k_t^e + \eta(k_t^e, e_{t+1}) \epsilon\, \mathsf{y}_{\text{eq}}^b(k_t^e, e_{t+1})}{w(k_t^e, e_{t+1})} \tag{A.58}$$

for all $(k_t^e, e_{t+1}) \in \mathbb{R}_{++} \times \mathbb{R}_+$. The claim now follows from the fact that for all $0 \leq k_t^e \leq \frac{1}{1+n} w^d(k_t, e_t)$, the r.h.s. in (A.58) is larger than the r.h.s. in (A.57). □

Proof of Proposition 3. Let $(k_t, e_t) \in \mathbb{R}_{++} \times \mathbb{R}_+$ and $w_t^d = w^d(k_t, e_t) > 0$ be arbitrary but fixed and consider the function

$$\mathcal{E}(k^e, k_t, e_t) := \tfrac{1}{1+n} s\big(w^d(k_t, e_t), d\big(k^e, E_{\text{eq}}(k_t, e_t)\big), R\big(k^e, E_{\text{eq}}(k_t, e_t)\big)\big) - k^e.$$

A correct forecast is a solution $0 \le k_t^e \le \frac{1}{1+n} w_t^d$ to

$$\mathcal{E}(k_t^e, k_t, e_t) \overset{!}{=} 0. \tag{A.59}$$

For better readability, set $e_{t+1} = E_{\text{eq}}(k_t, e_t)$. To establish the existence of k_t^e, we use the intermediate-value theorem. Since savings are bounded from above by w_t^d, we have

$$\mathcal{E}(\tfrac{w_t^d}{1+n}, k_t, e_t) \le 0.$$

By Assumption 1, we have $f_g(0) = f_b(0) = 0$ so that $d(0, e_{t+1}) = 0$. Since old-age consumption is ordinary, $R \mapsto Rs(w^d, d, R) + d$ is increasing for each w^d and each d. Consequently,

$$\begin{aligned} &\lim_{k^e \to 0} \Big[R(k^e, e_{t+1}) s\big(w_t^d, d(k^e, e_{t+1}), R(k, e_{t+1})\big) + d(k^e, e_{t+1}) \Big] \\ = &\lim_{k^e \to 0} R(k^e, e_{t+1}) s\big(w_t^d, 0, R(k^e, e_{t+1})\big) \ge \bar{c} \ > 0. \end{aligned}$$

Moreover, Assumption 1 implies $\lim_{k^e \to 0} R(k^e, e_{t+1}) k^e = 0$. It follows that

$$\begin{aligned} \lim_{k^e \to 0} \frac{s\big(w_t^d, d(k^e, e_{t+1}), R(k^e, e_{t+1})\big)}{k^e} &= \lim_{k^e \to 0} \frac{R(k^e, e_{t+1}) s\big(w_t^d, d(k^e, e_{t+1}), R(k^e, e_{t+1})\big)}{R(k^e, e_{t+1}) k^e} \\ &\ge \lim_{k^e \to 0} \frac{\bar{c}}{R(k^e, e_{t+1}) k^e} = \infty. \end{aligned}$$

As a consequence, in a sufficiently small neighbourhood $(0, \epsilon)$ of the origin, we have

$$\mathcal{E}(k^e, k_t, e_t) > 0 \quad \text{for all} \quad k^e \in (0, \epsilon).$$

The existence of a solution $0 < k_t^e \le \frac{w_t^d}{1+n}$ to (A.59) now follows from the intermediate-value theorem. Uniqueness of k_t^e is established next. By Assumption 2, $d \mapsto s(w_t^d, d, R)$ is strictly decreasing for all $R > 0$. By Assumption 1, $k^e \mapsto R(k^e, e_{t+1})$ is non-increasing. If, in addition, $R \mapsto s(w_t^d, d, R)$ is non-decreasing for all d and $k^e \mapsto d(k^e, e_{t+1})$ is non-decreasing, then $k^e \mapsto \mathcal{E}(k^e, k_t, e_t)$ is strictly decreasing and the solution k_t^e uniquely determined.

Since $(k_t, e_t) \in \mathbb{R}_{++} \times \mathbb{R}_+$ was arbitrary, it follows that there exists a forecasting rule G which is perfect in the sense of Definition 3. Note that if $k_t = 0$, then $w^d(0, e_t) = 0$ so that $k_t^e = 0$ is the unique solution to (A.59). □

Proof of Theorem 3. Let an arbitrary optimal allocation $\big\{(k_t^*, e_t^*, y_t^{b*}, c_t^{1*}, c_t^{2*})\big\}_{t=0}^{\infty}$ with corresponding Lagrange multipliers $\big\{(\lambda_t^{1*}, \lambda_t^{2*})\big\}_{t=0}^{\infty}$ be given, and consider some arbitrary but fixed period $t \ge 0$. Using the function Ω given in Lemma 1, denote the optimal wage-rental ratio by

$$\omega_t^* := \Omega\big(k_t^*, y_*^b(k_t^*, e_t^*)\big).$$

The proof proceeds by induction. Suppose that the current state of the economy is optimal, i.e., $(k_t, e_t) = (k_t^*, e_t^*)$.

Step 1 (Emissions tax rate). There exist thresholds $\eta_1, \eta_2 \in \mathbb{R}$ such that

$$\Omega_{\text{eq}}(k_t^*, \eta) = \Omega_b(k_t^*) \quad \text{for all } \eta \leq \eta_1 \qquad \text{and} \qquad \Omega_{\text{eq}}(k_t^*, \eta) = \Omega_g(k_t^*) \quad \text{for all } \eta \geq \eta_2.$$

Hence, there exists an optimal emissions tax rate $\eta_t^* \in [\eta_1, \eta_2]$ such that

$$\Omega_{\text{eq}}(k_t^*, \eta_t^*) = \omega_t^*. \tag{A.60}$$

η_t^* is unique since $\eta \mapsto \Omega_{\text{eq}}(k_t^*, \eta)$ is monotonic. If the government implements η_t^*, then the production plan in period t is optimal because (A.60) and (4.4) imply that

$$y_t^b = \mathsf{y}_{\text{eq}}^b(k_t, e_t) = \mathsf{y}_{\text{eq}}^b(k_t^*, e_t^*) = y_t^{b*}.$$

Consequently, the pollution stock at the beginning of $t+1$ is optimal

$$e_{t+1} = E\big(k_t, \mathsf{y}_{\text{eq}}^b(k_t, e_t)\big) = E(k_t^*, y_t^{b*}) = e_{t+1}^*. \tag{A.61}$$

Step 2 (Gross return on capital). Using Lemma 1, the first-order condition for an optimal policy (3.9) can be written as

$$1 - \varrho(\omega_t^*) + \lambda_t^{1*} - \lambda_t^{2*} = \frac{\epsilon(1 + \frac{\beta}{\gamma})}{u'(c_t^{1*})} \sum_{j=t}^{\infty} \left[\tfrac{\gamma(1-\zeta)}{1+n}\right]^{j-t} \mu'\big((1+n)e_{j+1}^*\big). \tag{A.62}$$

It follows from (A.62) that $\omega_t^* = \Omega_g(k_t^*)$ and $\lambda_t^{1*} \geq 0$, $\lambda_t^{2*} = 0$ if and only if

$$\varrho(\Omega_g(k_t^*)) \geq 1 - \frac{\epsilon(1 + \frac{\beta}{\gamma})}{u'(c_t^{1*})} \sum_{j=t}^{\infty} \left[\tfrac{\gamma(1-\zeta)}{1+n}\right]^{j-t} \mu'\big((1+n)e_{j+1}^*\big), \tag{A.63}$$

whereas $\omega_t^* = \Omega_b(k_t^*)$ and $\lambda_t^{2*} \geq 0$, $\lambda_t^{1*} = 0$ if and only if

$$\varrho(\Omega_b(k_t^*)) \leq 1 - \frac{\epsilon(1 + \frac{\beta}{\gamma})}{u'(c_t^{1*})} \sum_{j=t}^{\infty} \left[\tfrac{\gamma(1-\zeta)}{1+n}\right]^{j-t} \mu'\big((1+n)e_{j+1}^*\big). \tag{A.64}$$

It follows from (A.60), (A.63), (A.64), and Proposition 2 that

$$\eta_t^* = \frac{(1 + \frac{\beta}{\gamma})}{u'(c_t^{1*})} \sum_{j=t}^{\infty} \left[\tfrac{\gamma(1-\zeta)}{1+n}\right]^{j-t} \mu'\big((1+n)e_{j+1}^*\big). \tag{A.65}$$

Inserting (A.65), the first-order condition (A.62) takes the form

$$1 - \epsilon\eta_t^* + \lambda_t^{1*} - \lambda_t^{2*} = \varrho(\omega_t^*). \tag{A.66}$$

We next show that
$$R(k_t^*, e_t^*) = \frac{\partial f}{\partial k}(k_t^*, y_t^{b*}) + \lambda_t^{2*} f_b'(\kappa_b(\omega_t^*)). \tag{A.67}$$
From (4.3) we know that
$$R(k_t^*, e_t^*) = 1 - \delta + \max\Big\{ f_g'(\kappa_g(\omega_t^*)), [1 - \epsilon\eta_t^*] f_b'(\kappa_b(\omega_t^*)) \Big\},$$
and from Lemma 1 that
$$\frac{\partial f}{\partial k}(k_t^*, y_t^{b*}) = 1 - \delta + f_g'(\kappa_g(\omega_t^*)). \tag{A.68}$$
It follows from (A.66) that
$$\lambda_t^{1*} \geq 0, \lambda_t^{2*} = 0 \quad \iff \quad f_g'(\kappa_g(\omega_t^*)) \geq [1 - \epsilon\eta_t^*] f_b'(\kappa_b(\omega_t^*)).$$
Hence, (A.67) holds if $\omega_t^* \neq \Omega_b(k_t^*)$. Moreover, (A.66) implies
$$\omega_t^* = \Omega_b(k_t^*), \lambda_t^{1*} = 0, \lambda_t^{2*} > 0 \quad \iff \quad f_g'(\kappa_g(\omega_t^*)) < [1 - \epsilon\eta_t^*] f_b'(\kappa_b(\omega_t^*)).$$
In this case,
$$R(k_t^*, e_t^*) = 1 - \delta + [1 - \epsilon\eta_t^*] f_b'(\kappa_b(\omega_t^*)) \tag{A.69}$$
and rearranging (A.66) yields
$$\lambda_t^2 f_b'(\kappa_b(\omega_t^*)) = [1 - \epsilon\eta_t^*] f_b'(\kappa_b(\omega_t^*)) - f_g'(\kappa_g(\omega_t^*)). \tag{A.70}$$
Inserting (A.68) – (A.70) shows that (A.67) holds if $\omega_t^* = \Omega_b(k_t^*)$.

Substituting (A.67), the social planner's first-order condition (3.8) takes the form
$$\frac{u'(c_t^{1*})}{\beta u'(c_{t+1}^{2*})} = R(k_{t+1}^*, e_{t+1}^*). \tag{A.71}$$

Step 3 (Transfers). We next show that there exist feasible intergenerational transfers. The feasibility property of the optimal allocation implies
$$\begin{aligned} k_{t+1}^* &= A(k_t^*, y_t^{b*}, c_t^{1*}, c_t^{2*}) \\ &= \tfrac{1}{1+n}\Big[f(k_t^*, y_t^{b*}) - c_t^{1*} - \tfrac{c_t^{2*}}{1+n} \Big] \\ &= \tfrac{1}{1+n}\Big[w(k_t^*, e_t^*) + R(k_t^*, e_t^*) k_t^* + \eta_t^* \epsilon y_t^{b*} - c_t^{1*} - \tfrac{c_t^{2*}}{1+n} \Big] > 0, \end{aligned} \tag{A.72}$$
where the last equation follows from the balance equation (4.5). Let
$$d_t := c_t^{2*} - R(k_t^*, e_t^*)(1+n) k_t^* \tag{A.73}$$
be the old-age transfer, and the corresponding lump-sum transfer to the young generation

be

$$a_t = \eta_t^* \epsilon y_t^{b*} - \tfrac{d_t}{1+n}. \tag{A.74}$$

Using (A.73) and (A.74), it follows from (A.72) that

$$c_t^{1*} = w(k_t^*, e_t^*) + \eta_t^* \epsilon y_t^{b*} - \tfrac{d_t}{1+n} - (1+n)k_{t+1}^* = w(k_t^*, e_t^*) + a_t - (1+n)k_{t+1}^* \tag{A.75}$$

$$c_{t+1}^{2*} = d_{t+1} + R(k_{t+1}^*, e_{t+1}^*)(1+n)k_{t+1}^*. \tag{A.76}$$

Conditions (A.71), (A.75), and (A.76) coincide with the optimality conditions for an individual savings decision of a young agent who has the disposable income $w(k_t^*, e_t^*) + a_t$ and correctly anticipates the return $R(k_{t+1}^*, e_{t+1}^*)$ and the old-age transfer payment d_{t+1}. Hence,

$$k_{t+1}^* = \tfrac{1}{1+n} s\big(w(k_t^*, e_t^*) + a_t, d_{t+1}, R(k_{t+1}^*, e_{t+1}^*)\big) \tag{A.77}$$

so that under perfect foresight, $k_{t+1} = k_{t+1}^*$, $c_t^1 = c_t^{1*}$, and $c_{t+1}^2 = c_{t+1}^{2*}$.

To translate a_t into an income tax rate τ_t^*, set

$$\tau_t^* := -\frac{a_t}{w(k_t^*, e_t^*)}. \tag{A.78}$$

Inserting (A.72) – (A.74) into (A.78), it follows that

$$\tau_t^* = -\frac{R(k_t^*, e_t^*)k_t^* + \eta_t^* \epsilon y_t^{b*} - \frac{c_t^{2*}}{1+n}}{w(k_t^*, e_t^*)} = \frac{w(k_t^*, e_t^*) - c_t^{1*} - (1+n)k_{t+1}^*}{w(k_t^*, e_t^*)}. \tag{A.79}$$

Since the allocation is optimal, it satisfies $k_{t+1}^* > 0$ and $c_t^{1*}, c_t^{2*} > 0$, so that it follows from (A.79) that $\underline{\tau}(k_t^*, \eta_t^*) < \tau_t^* < 1$. Hence, the fiscal policy (η_t^*, τ_t^*) is feasible. Since $(k_t^*, e_t^*) \in \mathbb{R}_{++} \times \mathbb{R}_+$ was arbitrary, it is feasible in the sense of Definition 2.

Since

$$w(k_t^*, e_t^*) + a_t = (1 - \tau_t^*)w(k_t^*, e_t^*) = w^d(k_t^*, e_t^*)$$

and

$$d_{t+1} = (1+n)\big[\eta_{t+1}^* \epsilon y_{t+1}^{b*} + \tau_{t+1}^* w(k_{t+1}^*, e_{t+1}^*)\big] = d(k_{t+1}^*, e_{t+1}^*),$$

(A.77) takes the form

$$k_{t+1}^* = \tfrac{1}{1+n} s\big(w^d(k_t^*, e_t^*), d(k_{t+1}^*, e_{t+1}^*), R(k_{t+1}^*, e_{t+1}^*)\big).$$

Since the period t was arbitrary, induction implies that there exists a series of feasible tax rates $\{(\eta_t^*, \tau_t^*\}_{t=0}^{\infty}$ such that $\big\{(k_t^*, e_t^*, y_t^{b*}, c_t^{1*}, c_t^{2*})\big\}_{t=0}^{\infty}$ is a perfect-foresight allocation.

It remains to establish the optimal tax-policy rules. It follows directly from (A.60) and (A.79) that the tax rates η_t^* and τ_t^* depend on the current state only (k_t^*, e_t^*). Since the initial conditions $(k_0, e_0) \in \mathbb{R}_{++} \times \mathbb{R}_+$ are arbitrary, there must exist tax-policy rules of the form

$$\eta_*, \tau_* : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}$$

such that $\eta_t^* = \eta_*(k_t^*, e_t^*)$ and $\tau_t^* = \tau_*(k_t^*, e_t^*)$ for each $t \geq 0$. $\square$

Proof of Corollary 3. We prove the corollary for a more general case. Consider an arbitrary steady state $(\bar{k}, \bar{y}^b) \in \mathbb{R}_{++} \times \mathbb{R}_+$ with $0 \leq \bar{y}^b \leq f_b(\bar{k})$ and $\bar{e} = \frac{\epsilon}{n+\zeta}\bar{y}^b$. In particular, this can be the modified golden-rule steady state. Observe that any feasible steady state must satisfy $(1+n)\bar{k} < f(\bar{k}, \bar{y}^b)$. By Proposition 2, there exist thresholds $\eta_1, \eta_2 \in \mathbb{R}$ such that

$$\Omega_{\mathrm{eq}}(\bar{k}, \eta) = \Omega_b(\bar{k}) \quad \text{for all } \eta \leq \eta_1 \qquad \text{and} \qquad \Omega_{\mathrm{eq}}(\bar{k}, \eta) = \Omega_g(\bar{k}) \quad \text{for all } \eta \geq \eta_2.$$

Hence, there exists $\bar{\eta} \in [\eta_1, \eta_2]$ such that $\Omega_{\mathrm{eq}}(\bar{k}, \bar{\eta}) = \Omega(\bar{k}, \bar{y}^b)$. Given $\bar{\eta}$, the factor prices are $\bar{w} = w(\bar{k}, \bar{e})$ and $\bar{R} = R(\bar{k}, \bar{e})$. Observe that

$$\underline{\tau}(\bar{k}, \bar{\eta}) < 1 \iff -\frac{\bar{R}\bar{k} + \bar{\eta}\epsilon\bar{y}^b}{\bar{w}} < 1 \iff \bar{w} + \bar{R}\bar{k} + \bar{\eta}\epsilon\bar{y}^b > 0.$$

Since the balance equation (4.5) implies $\bar{w} + \bar{R}\bar{k} + \bar{\eta}\epsilon\bar{y}^b = f(\bar{k}, \bar{y}^b) > 0$, it follows that $\underline{\tau}(\bar{k}, \bar{\eta}) < 1$. Thus, the range of feasible steady-state income tax rates is well defined by $\big[\underline{\tau}(\bar{k}, \bar{\eta}), 1\big]$. Observe that for the modified golden-rule steady state, it follows from (4.13) that

$$\bar{\eta} = \frac{1 + \frac{\beta}{\gamma}}{1 + \frac{\gamma(1-\zeta)}{1+n}} \frac{\mu'((1+n)\bar{e}_\gamma)}{u'(\bar{c}_\gamma^1)} = \frac{\psi(\bar{k}_\gamma, \bar{y}_\gamma^b)}{\epsilon} =: \bar{\eta}_\gamma.$$

Theorem 2 now implies that

$$\begin{aligned} R(\bar{k}_\gamma, \bar{e}_\gamma) &= 1 - \delta + \max\Big\{ f_g'\big(\kappa_g(\Omega_{\mathrm{eq}}(\bar{k}_\gamma, \bar{\eta}_\gamma))\big), \big[1 - \epsilon\bar{\eta}_\gamma\big] f_b'\big(\kappa_b(\Omega_{\mathrm{eq}}(\bar{k}_\gamma, \bar{\eta}_\gamma))\big)\Big\} \\ &= 1 - \delta + \max\Big\{ f_g'\big(\kappa_g(\Omega_{\mathrm{eq}}(\bar{k}_\gamma, \bar{\eta}_\gamma))\big), \big[1 - \psi(\bar{k}_\gamma, \bar{y}_\gamma^b)\big] f_b'\big(\kappa_b(\Omega_{\mathrm{eq}}(\bar{k}_\gamma, \bar{\eta}_\gamma))\big)\Big\} \\ &= \tfrac{1+n}{\gamma}. \end{aligned}$$

It follows from Condition (4.11) that given the optimal steady-state emissions tax rate $\bar{\eta}$, the optimal steady-state income tax rate $\bar{\tau}$ is determined by a solution to

$$(1+n)\bar{k} \stackrel{!}{=} s\big((1-\tau)\bar{w}, (1+n)[\bar{\eta}\epsilon\bar{y}^b + \tau\bar{w}], \bar{R}\big) =: h(\tau). \tag{A.80}$$

Observe that Condition (A.80) is nothing but (4.16). Using the intermediate-value theorem, its existence is established next. Since savings are bounded from above by the disposable income, we have

$$h(1) = s(0, (1+n)f(\bar{k}, \bar{y}^b), \bar{R}) = 0.$$

On the other hand, by construction of the lower bound $\underline{\tau}(\bar{k}, \bar{\eta})$,

$$h\big(\underline{\tau}(\bar{k}, \bar{\eta})\big) = s\big(f(\bar{k}, \bar{y}^b), -(1+n)\bar{R}\bar{k}, \bar{R}\big). \tag{A.81}$$

The savings function on the r.h.s. of (A.81) solves the first-order condition

$$\frac{u'\Big(f(\bar{k},\bar{y}^b) - s\big(f(\bar{k},\bar{y}^b), -(1+n)\bar{R}\bar{k}, \bar{R}\big)\Big)}{\beta u'\Big(\bar{R}\Big[s\big(f(\bar{k},\bar{y}^b), -(1+n)\bar{R}\bar{k}, \bar{R}\big) - (1+n)\bar{k}\Big]\Big)} = \bar{R}. \tag{A.82}$$

Due to the Inada condition on u, the agent chooses strictly positive amounts of both youthful and old-age consumption. Therefore,

$$(1+n)\bar{k} < s\big(f(\bar{k},\bar{y}^b), -(1+n)\bar{R}\bar{k}, \bar{R}\big) < f(\bar{k},\bar{y}^b).$$

It follows that

$$h(1) < (1+n)\bar{k} < h\big(\underline{\tau}(\bar{k},\bar{\eta})\big),$$

so that a solution $\bar{\tau} \in \big(\underline{\tau}(\bar{k},\bar{\eta}), 1\big)$ to (A.80) exists. Since Assumption 2 implies $h' < 0$, the solution $\bar{\tau}$ is uniquely determined. □

Proof of Proposition 4. If the marginal damage is a constant $d > 0$, then (3.9) reads

$$\frac{\partial f}{\partial y^b}(k_t^*, y_t^{b*}) + \lambda_t^1 - \lambda_t^2 = \frac{\epsilon(1+\frac{\beta}{\gamma})}{1 - \frac{\gamma(1-\zeta)}{1+n}} \frac{d}{u'(c_t^{1*})}. \tag{A.83}$$

In a local neighbourhood of a mixed modified golden-rule steady state, we have $0 < y_t^{b*} < f_b(k_t^*)$ such that $\lambda_t^1 = \lambda_t^2 = 0$, $t \geq 0$. It follows from (3.7), (3.8), and (A.83) that

$$\frac{\dfrac{\partial f}{\partial y^b}(k_{t+1}^*, y_{t+1}^{b*})}{\dfrac{\partial f}{\partial y^b}(k_t^*, y_t^{b*})} = \tfrac{\gamma}{1+n}\frac{\partial f}{\partial k}(k_{t+1}^*, y_{t+1}^{b*}). \tag{A.84}$$

Inserting the partial derivatives of f given in Lemma 1 into (A.84) yields

$$\frac{1-\varrho(\omega_{t+1}^*)}{1-\varrho(\omega_t^*)} = \tfrac{\gamma}{1+n}\Big[1 - \delta + f_g'(\kappa_g(\omega_{t+1}^*))\Big]. \tag{A.85}$$

By Proposition 2, $1 - \varrho(\omega_t^*) = \epsilon\eta_t^*$ so that (A.85) takes the form (4.20). □

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